

Heath-Jarrow-Morton-Musiela equation with Lévy perturbation*

Michał Barski

Faculty of Mathematics, Cardinal Stefan Wyszyński University in Warsaw, Poland

Faculty of Mathematics and Computer Science, University of Leipzig, Germany

Michal.Barski@math.uni-leipzig.de

Jerzy Zabczyk

Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland

zabczyk@impan.pl

December 16, 2015

Abstract

The paper studies the Heath-Jarrow-Morton-Musiela equation of the bond market. The equation is analyzed in weighted spaces of functions defined on $[0, +\infty)$. Sufficient conditions for local and global existence are obtained. For equation with the linear diffusion term the conditions for global existence are close to the necessary ones.

Contents

1	Introduction	2
2	Preliminaries	4
I	HJMM equation with general diffusion	8
3	Formulation of the main results	8
4	Proofs of the results	11
4.1	Proof of Theorem 3.1	12
4.2	Proofs of Theorem 3.2, Theorem 3.7 and Theorem 3.4, Theorem 3.9	13
4.2.1	Local Lipschitzianity and linear growth of the coefficients in $L^{2,\gamma}$	13
4.2.2	Local Lipschitzianity and linear growth of the coefficients in $H^{1,\gamma}$	14
4.3	Proof of Theorem 4.1	16
4.4	Proof of Theorem 4.2	17

*Supported by The Polish MNiSW grant NN201419039.

II	HJMM equation with linear diffusion	19
5	Formulation of the main results	20
6	Proofs of the equivalence results	24
6.1	Equations in natural and moving frames	24
6.2	Proof of Theorem 5.9	25
6.3	Proof of Theorem 5.10	26
6.3.1	Step 1. Regularity of the random factor of (5.5)	26
6.3.2	Step 2. A priori regularity of the solution	29
7	Proofs of necessary conditions for existence in $H_+^{1,\gamma}$	31
8	Proofs of existence of global and strong solutions	31
9	Proof of the uniqueness of the solutions in $H_+^{1,\gamma}$	35
10	Appendix	37
10.1	HJM approach to the bond market	37
10.2	Laplace exponent	38

1 Introduction

The Heath-Jarrow-Morton-Musiela equation, driven by a real Lévy process L , is a stochastic partial differential equation of the form

$$dr(t, x) = \left[\frac{d}{dx} r(t, x) + F(r(t))(x) \right] dt + G(r(t-))(x) dL(t),$$

$$r(0, x) = r_0(x), \quad x \geq 0, \quad t \in (0, T^*], \quad (1.1)$$

where the diffusion operator G and the drift F are of the form:

$$G(r)(x) := g(x, r(x)); \quad F(r)(x) := J' \left(\int_0^x g(v, r(v)) dv \right) g(x, r(x)). \quad (1.2)$$

The function J' admits a representation

$$J'(z) = -a + qz + \int_{\mathbb{R}} y(\mathbf{1}_{(-1,1)}(y) - e^{-zy}) \nu(dy), \quad z \in \mathbb{R},$$

with $a \in \mathbb{R}$, $q \geq 0$ and the measure ν satisfies the following integrability condition

$$\int_{\mathbb{R}} (y^2 \wedge 1) \nu(dy) < \infty.$$

The measure ν is the Lévy measure of the process L . The function g has a financial meaning and is sometimes called volatility of the bond market. Solutions to (1.1) are, the so called, forward curves, see e.g. [7], [6], [14] and the quantity

$$P(t, T) = e^{-\int_0^{T-t} r(t, v) dv}, \quad t \leq T,$$

can be interpreted as the price, at moment t , of the bond which matures at moment T (and then pays 1). The equation (1.1) describes the dynamics of the forward curves in the *moving frame* and was introduced by Musiela in [13]. The original version, in the *natural frame*, appeared first in the PhD dissertation of Morton [12]. For more information about the financial background of the equation see Appendix 10.1.

If the process L is not present in the equation, that is if L is identically zero, then $J' = 0$, $F = 0$ and $G = 0$ and the equation has trivial solution $r(t, x) = r_0(t + x)$. So only the stochastic case is of real interest. The equation was intensively studied in the case when L is a Wiener process, see e.g. [6], [14] and references therein. Then the function J' is linear: $J'(z) = qz$, $z \in \mathbb{R}$, $q \geq 0$. There are also several results for the case of general infinite dimensional Lévy process L , see e.g. [14], [5], [17], [18], [19], [10], [8], [15]. In particular in [10] local solvability of (1.1) was studied for Lévy process L having exponential moments, the assumption which we find very restrictive. In fact a majority of the results presented in the paper can be extended to infinite dimensional noise. Our intention was to obtain optimal results in the most important case of the one dimensional process L , to see what kind of results could be expected in the general case.

The aim of the present paper is to establish existence and uniqueness of weak solutions to (1.1). We restrict our attention to positive solutions which are relevant for applications. The equation is studied either in the Hilbert space $H = L^{2,\gamma}$, of square summable functions h on $[0, +\infty)$ with the norm

$$\|h\|_{L^{2,\gamma}} := \left(\int_0^{+\infty} |h(x)|^2 e^{\gamma x} dx \right)^{\frac{1}{2}} < +\infty, \quad (1.3)$$

or in the Hilbert space $H = H^{1,\gamma}$, of absolutely continuous functions h on $[0, +\infty)$ such that

$$\|h\|_{H^{1,\gamma}} := \left(\int_0^{+\infty} (|h(x)|^2 + |h'(x)|^2) e^{\gamma x} dx \right)^{\frac{1}{2}} < +\infty, \quad (1.4)$$

with $\gamma > 0$. Similar results can be obtained for spaces with different weight functions.

The paper is divided into Part I and Part II. Part I studies equation (1.1) with general volatility g and uses some versions of the contraction mapping theorem. Part II is devoted to the case when g is a linear function of the second variable:

$$g(x, y) = \lambda(x)y, \quad x, y \geq 0.$$

In the latter case, more special but important, better results can be obtained using some monotonicity properties of the equation.

Part I starts with formulating local and global existence results in the sets $L_+^{2,\gamma}$ and $H_+^{1,\gamma}$ of positive functions in $L^{2,\gamma}$ and $H^{1,\gamma}$ respectively, see Theorem 3.2, Theorem 3.7 and Theorem 3.4, Theorem 3.9. The main tool here is some extension to locally Lipschitz coefficients of the standard result on existence of positive solutions to stochastic evolution equations. The proofs start from establishing first abstract existence results and then proving local Lipschitz properties

and linear growth of the coefficients as well as checking conditions for positivity. It turns out that only for restrictive class of functions J' the diffusion G and the drift F can be locally Lipschitz or of linear growth.

Part II is devoted to the equation

$$dr(t, x) = \left[\frac{d}{dx} r(t, x) + J' \left(\int_0^x \lambda(v) r(t, v) dv \right) \lambda(x) r(t, x) \right] dt + \lambda(x) r(t-, x) dL(t),$$

$$r(0, x) = r_0(x), \quad x \geq 0, \quad t \in (0, T^*]. \quad (1.5)$$

where $\lambda(\cdot)$ is a continuous, positive and bounded function. From results of Part I one deduces easily sufficient conditions for existence of local, positive solutions to (1.5). They are formulated as Theorem 5.1 and Theorem 5.2. Main results on existence of global solutions are presented as Theorem 5.3 and Theorem 5.5. Uniqueness is proved in Theorem 5.8. Moreover, Theorem 5.7 gives conditions under which global solutions are strong. The proofs of those results exploit the fact that the weak form of the equation (1.5) is equivalent to the equation

$$r(t, x) = a(t, x) e^{\int_0^t J'(\int_0^{t-s+x} \lambda(v) r(s, v) dv) \lambda(t-s+x) ds}, \quad x \geq 0, \quad t \in (0, T^*], \quad (1.6)$$

where

$$a(t, x) := r_0(t+x) e^{\int_0^t \lambda(t-s+x) dL(s) - \frac{q^2}{2} \int_0^t \lambda^2(t-s+x) ds}$$

$$\cdot \prod_{0 \leq s \leq t} (1 + \lambda(t-s+x) \Delta L(s)) e^{-\lambda(t-s+x) \Delta L(s)}.$$

The equivalence of (1.6) and the weak form of (1.5) is established in Section 5, in Theorem 5.9 and Theorem 5.10, preceding the proofs of the main results. The proofs are rather involved and require some new results on regularity of Lévy fields of independent interest, see Proposition 6.2 Proposition 6.3. Standard methods exploiting the Lipschitzianity of the coefficients, applied for instance in [8], [10], require more restrictive conditions.

As we have already said, the study of the linear HJMM equation was initiated by Morton, in his PhD dissertation [12]. He showed that the equation (1.5) in the *natural frame*, with L being a Wiener process, does not have a solution in the class of bounded functions of two arguments on a finite domain. The situation changes substantially when L is a general Lévy process and results on existence and explosions for the equation (1.5) but in the *natural frame* were obtained [1].

The present paper is a much elaborated version of the note [2] presented in arxiv.

Acknowledgements. The authors would like to thank S. Peszat and A. Rusinek for inspiring discussions on the subject of the paper.

2 Preliminaries

We gather first results on properties of the *Laplace exponent* J of the process L and its derivatives, which will be frequently used in the following sections of the paper. The first derivative J' ,

appears explicitly in the basic equation (1.1)-(1.2). As our prime issue will be the solvability of (1.1)-(1.2) in the set of non-negative functions we concentrate on the properties of J and its derivatives for non-negative arguments.

The function J is defined by

$$\mathbf{E}(e^{-zL(t)}) = e^{tJ(z)}, \quad t \in [0, T^*], \quad z \in \mathbb{R}, \quad (2.1)$$

and admits explicit representation

$$J(z) = -az + \frac{1}{2}qz^2 + \int_{\mathbb{R}} (e^{-zy} - 1 + zy\mathbf{1}_{(-1,1)}(y)) \nu(dy), \quad z \in \mathbb{R}, \quad (2.2)$$

with $a \in \mathbb{R}$, $q \geq 0$ and the measure ν satisfies the following integrability condition

$$\int_{\mathbb{R}} (y^2 \wedge 1) \nu(dy) < \infty. \quad (2.3)$$

It is easy to see that J is well defined for all positive numbers z if and only if

$$\int_{-\infty}^{-1} e^{zy} \nu(dy) < +\infty, \quad z \geq 0.$$

Its derivative, J' is of the form

$$J'(z) = -a + qz + \int_{\mathbb{R}} y(\mathbf{1}_{(-1,1)}(y) - e^{-zy}) \nu(dy), \quad z \in \mathbb{R}. \quad (2.4)$$

It is clear that $|J'(0)| < +\infty$ if and only if

$$(B0) \quad \int_{|y|>1} |y| \nu(dy) < +\infty,$$

and $|J'(z)| < +\infty, z > 0$ iff

$$\int_{-\infty}^{-1} |y| e^{zy} \nu(dy) < +\infty.$$

In particular, if the support of the Lévy measure is bounded from below then J' is well defined and continuous on $[0, +\infty)$ if (B0) is satisfied. J' is automatically increasing on its domain and its derivative is equal to:

$$J''(z) = q + \int_{\mathbb{R}} y^2 e^{-zy} \nu(dy), \quad z \in \mathbb{R}. \quad (2.5)$$

The results on the function J' formulated below are explained in detail in Section 10.2 in Appendix. Note that the behavior of J' near the origin depends on the behavior of ν on $[-1, 1]^c$.

Proposition 2.1 *The function J' is Lipschitz on $[0, z_0]$, $z_0 > 0$ if and only if*

$$(L1) \quad \int_{-\infty}^{-1} |y|^2 e^{z_0|y|} \nu(dy) < +\infty, \quad \text{and} \quad \int_1^{+\infty} y^2 \nu(dy) < +\infty.$$

The function J'' is Lipschitz on $[0, z_0]$, $z_0 > 0$ if and only if

$$(L2) \quad \int_{-\infty}^{-1} |y|^3 e^{z_0|y|} \nu(dy) < +\infty, \quad \text{and} \quad \int_1^{+\infty} y^3 \nu(dy) < +\infty.$$

Proposition 2.2 *The function J' is bounded on $[0, +\infty)$ iff*

$$(B1) \quad q = 0, \quad \text{supp}\{\nu\} \subseteq [0, +\infty) \quad \text{and} \quad \int_0^{+\infty} y\nu(dy) < +\infty.$$

The function J'' is bounded on $[0, +\infty)$ iff

$$(B2) \quad \text{supp}\{\nu\} \subseteq [0, +\infty) \quad \text{and} \quad \int_1^{\infty} y^2\nu(dy) < \infty.$$

In the second part of the paper we will need more involved assumptions on the growth of the function J' .

$$(B3) \quad \text{For some } a > 0, b \in \mathbb{R}, \quad J'(z) \geq a(\ln z)^3 + b, \quad \text{for all } z > 0.$$

$$(B4) \quad \limsup_{z \rightarrow \infty} (\ln z - \bar{\lambda} T^* J'(z)) = +\infty, \quad 0 < T^* < +\infty;$$

If J' is a bounded function then (B4) obviously holds. Thus, in particular, (B4) is satisfied for subordinators (increasing Lévy processes) with possible drifts, see Proposition 4.1 in [1]. However, (B4) does not imply that J' is bounded, see Example 4.2 in [1]. Moreover, we have the following result, see [1].

Proposition 2.3 *If $q > 0$ or $\nu\{(-b, 0)\} > 0$, $b > 0$ in the representation (2.2), then J' satisfies (B3).*

This means that each Lévy process with non-degenerate Wiener part or negative jumps automatically satisfies (B3). Moreover, if L does not have the Wiener part nor negative jumps then (B4) is affected only by the behavior of ν close to zero. To see this, note that

$$\sup_{z \geq 0} \int_1^{+\infty} y e^{-zy} \nu(dy) < +\infty,$$

which means that the part of J' corresponding to jumps greater than 1 is bounded. Thus (B4) in fact depends on the growth of the function

$$z \rightarrow \int_0^1 y e^{-zy} \nu(dy) < +\infty.$$

Below we formulate the conditions (B3) and (B4) explicitly in terms of the measure ν , for the proofs we refer to [1]. Let us recall that a positive function M *varies slowly at 0* if for any fixed $x > 0$

$$\frac{M(tx)}{M(t)} \rightarrow 1, \quad \text{as } t \rightarrow 0.$$

If

$$\frac{f(x)}{g(x)} \rightarrow 1, \quad \text{as } x \rightarrow 0,$$

we write $f(x) \sim g(x)$.

Proposition 2.4 *Assume that for some $\rho \in (0, +\infty)$,*

$$(B5) \quad \int_0^x y^2 \nu(dy) \sim x^\rho \cdot M(x), \quad \text{as } x \rightarrow 0,$$

where M is a slowly varying function at 0.

i) If $\rho > 1$ then (B_4) holds.

ii) If $\rho < 1$, then (B_3) holds.

iii) If $\rho = 1$, the measure ν has a density and

$$M(x) \longrightarrow 0 \quad \text{as } x \rightarrow 0, \quad \text{and} \quad \int_0^1 \frac{M(x)}{x} dx = +\infty, \quad (2.6)$$

then (B_4) holds.

Part I

HJMM equation with general diffusion

By classical results, see e.g. [14], existence of weak solution to (1.1) is equivalent to the existence of a solution to the integral version of (1.1):

$$\begin{aligned} r(t, x) = & S_t(r_0)(x) + \int_0^t S_{t-s}(F(r(s)))(x)ds \\ & + \int_0^t S_{t-s}(G(r(s-)))(x)dL(s), \quad x \geq 0, \quad t \in (0, T^*], \end{aligned} \quad (2.7)$$

called mild solution. In (2.7), $\{S_t, t \geq 0\}$, stands for the shift semigroup

$$S_t(h)(x) := h(t+x), \quad t \geq 0, \quad x \geq 0, \quad h \in H,$$

acting on the Hilbert space H . The equation (2.7) will be treated here within the standard SPDE framework for which the crucial role is played by the Lipschitz properties of the transformations F and G . Existence of positive solutions is deduced from abstract results presented in Section 4. Theorem 4.1 generalizes standard results on existence, see [14], to the case when coefficients have linear growth and are locally Lipschitz. To obtain positivity of solutions we use Theorem 4.2 which is a generalized version of the result of Milian, see [11], and provides if and only if conditions for positivity in the framework of locally Lipschitz coefficients. As a corollary, in Theorem 3.1 we obtain direct conditions for positivity in our model.

The results on existence of local solutions in $L_+^{2,\gamma}$ and $H_+^{1,\gamma}$ are formulated as Theorem 3.2 and Theorem 3.4, respectively. They require some regularity properties of the function g as well as local Lipschitz property for J' and J'' which in turn reduce to the integrability conditions (L1), (L2) for the Lévy measure on the complement of $[-1, 1]$. Theorem 3.3, and Theorem 3.5, which are reformulations of the above results, show that, in particular, local solutions exist for the noise with small jumps only and the Wiener process.

For the results on global solutions, which are formulated as Theorem 3.7 and Theorem 3.9, we need more assumptions. For the space $L_+^{2,\gamma}$ boundedness of J' on $[0, +\infty)$ is required and for $H_+^{1,\gamma}$ boundedness of both J' and J'' is needed. These conditions are rather restrictive and exclude all Lévy processes which have Wiener part or negative jumps, see Theorem 3.8 and Theorem 3.10.

Proofs are postponed to Section 4.

3 Formulation of the main results

We start from a general result on positivity of the solutions to the equation (2.7) which throws some light on the conditions imposed in the sequel. It is a consequence of our generalization of an abstract result on positivity due to Milian, see Theorem 4.2.

Theorem 3.1 Assume that G and F in (2.7) are locally Lipschitz in H . Then (2.7) is positivity preserving if and only if

$$\begin{aligned} r + g(x, r)u &\geq 0 && \text{for all } r \geq 0, x \geq 0, u \in \text{supp } \nu, \\ g(x, 0) &= 0 && \text{for all } x \geq 0. \end{aligned} \quad (3.1)$$

LOCAL EXISTENCE

For solvability of the HJMM equation in $L_+^{2,\gamma}$ we will need the following conditions on g :

$$(G1) \quad \left\{ \begin{array}{ll} (i) & \text{The function } g \text{ is continuous on } \mathbb{R}_+^2 \text{ and} \\ & g(x, 0) = 0, \quad g(x, y) \geq 0, \quad x, y \geq 0. \\ (ii) & \text{For all } x, y \geq 0 \text{ and } u \in \text{supp } \nu : \\ & x + g(x, y)u \geq 0. \\ (iii) & \text{There exists a constant } C > 0 \text{ such that} \\ & |g(x, u) - g(x, v)| \leq C |u - v|, \quad x, u, v \geq 0. \end{array} \right.$$

Theorem 3.2 Assume that J' satisfies Lipschitz condition in some interval $[0, z_0]$, $z_0 > 0$ and that (G1) holds. Then for arbitrary initial condition $r_0 \in L_+^{2,\gamma}$ there exists a unique local solution of (2.7) in $L_+^{2,\gamma}$.

In view of Proposition 2.1 we get more explicit result.

Theorem 3.3 Assume that (G1) holds and either L is a Wiener process or for some $z_0 > 0$:

$$\int_{-\infty}^{-1} |y|^2 e^{z_0|y|} \nu(dy) < +\infty, \quad \text{and} \quad \int_1^{+\infty} y^2 \nu(dy) < +\infty.$$

Then for arbitrary initial condition $r_0 \in L_+^{2,\gamma}$ there exists a unique local solution of (2.7) in $L_+^{2,\gamma}$.

For local existence in $H_+^{1,\gamma}$ we will need more stringent conditions on g :

$$(G2) \quad \left\{ \begin{array}{ll} (i) & \text{The functions } g'_x, g'_y \text{ are continuous on } \mathbb{R}_+^2 \text{ and} \\ & g'_x(x, 0) = 0, \quad x \geq 0. \\ (ii) & \sup_{x,y \geq 0} |g'_y(x, y)| < +\infty, \\ (iii) & \text{There exists a constant } C > 0 \text{ such that} \\ & |g'_x(x, u) - g'_x(x, v)| + |g'_y(x, u) - g'_y(x, v)| \leq C |u - v|, \quad x, u, v \geq 0. \end{array} \right.$$

Theorem 3.4 Assume that J' and J'' satisfy Lipschitz condition in some interval $[0, z_0]$, $z_0 > 0$ and that (G1) and (G2) hold. Then for arbitrary initial condition $r_0 \in H_+^{1,\gamma}$ there exists a unique local solution of (2.7) in $H_+^{1,\gamma}$.

From Proposition 2.1 and Proposition 2.2 one can deduce more explicit result.

Theorem 3.5 Assume that (G1) and (G2) hold and for some $z_0 > 0$

$$\int_{-\infty}^{-1} |y|^3 e^{z_0|y|} \nu(dy) < +\infty, \quad \text{and} \quad \int_1^{+\infty} y^3 \nu(dy) < +\infty.$$

Then for arbitrary initial condition $r_0 \in H_+^{1,\gamma}$ there exists a unique local solution of (2.7) in $H_+^{1,\gamma}$.

Some comments on the imposed conditions are in place now.

If $\text{supp}\{\nu\} \subseteq [0, +\infty)$, that is when L has positive jumps only, and (G1)(i) holds then the crucial positivity condition (G1)(ii) is satisfied. More general result is true.

Proposition 3.6 If for some $m \geq 0$, $\text{supp}\{\nu\} \subseteq [-m, +\infty)$ and (G1)(i) holds then the condition (G1)(ii) holds if and only if

$$0 \leq g(x, y) \leq \frac{y}{m}, \quad x, y \geq 0.$$

If $\bar{g}(y) := \sup_{x \geq 0} g(x, y) < +\infty$, then (G1)(ii) holds if and only if

$$\text{supp}\{\nu\} \subseteq \left[-\inf_{y \geq 0} \frac{y}{\bar{g}(y)}, +\infty \right).$$

Proof: Indeed, we have $x + g(x, y) \geq x - g(x, y)m \geq 0$.

Moreover (G1)(ii) holds iff for all $u \in \text{supp}\{\nu\}$

$$u \geq -\inf_{x, y \geq 0} \frac{y}{g(x, y)} = -\inf_{y \geq 0} \frac{y}{\bar{g}(y)}. \quad (3.2)$$

□

GLOBAL EXISTENCE

We pass now to the global existence results first in $L_+^{2,\gamma}$ and then in $H_+^{1,\gamma}$.

Theorem 3.7 Assume that J' is Lipschitz on some $[0, z_0]$, $z_0 > 0$ and bounded on $[0, +\infty)$ and that (G1) holds. Then for arbitrary $r_0 \in L_+^{2,\gamma}$ the equation (2.7) has unique global solution in $L_+^{2,\gamma}$.

In virtue of Proposition 2.1 and Proposition 2.2 we get more explicit result.

Theorem 3.8 Assume that (G1) holds and in addition:

$$q = 0, \quad \text{supp}\{\nu\} \subseteq [0, +\infty), \quad \int_0^{+\infty} \max\{y, y^2\} \nu(dy) < +\infty.$$

Then for arbitrary $r_0 \in L_+^{2,\gamma}$ the equation (2.7) has unique global solution in $L_+^{2,\gamma}$.

For global existence in $H_+^{1,\gamma}$ we need additional conditions on g :

$$(G3) \quad \begin{cases} (i) & \text{Partial derivatives } g'_y, g''_{xy}, g''_{yy} \text{ are bounded on } \mathbb{R}_+^2. \\ (ii) & 0 \leq g(x, y) \leq c\sqrt{y}, \quad x, y \geq 0, \\ (iii) & |g'_x(x, y)| \leq h(x), \quad x, y \geq 0, \text{ for some } h \in L_+^{2,\gamma}. \end{cases}$$

Theorem 3.9 *Let J', J'' be Lipschitz on some $[0, z_0]$, $z_0 > 0$ and bounded on $[0, +\infty)$. Assume that conditions (G1), (G2) and (G3) are satisfied. Then for arbitrary $r_0 \in H_+^{1,\gamma}$ there exists a unique global solution of (2.7) in $H_+^{1,\gamma}$.*

In virtue of Proposition 2.1 and Proposition 2.2 we get more explicit result.

Theorem 3.10 *Assume that conditions (G1), (G2) and (G3) are satisfied and*

$$q = 0, \quad \text{supp}\{\nu\} \subseteq [0, +\infty), \quad \int_0^{+\infty} \max\{y, y^3\} \nu(dy) < +\infty.$$

Then for arbitrary $r_0 \in H_+^{1,\gamma}$ there exists a unique global solution of (2.7) in $H_+^{1,\gamma}$.

4 Proofs of the results

The proofs will be based on general existence and positivity results for evolution equations:

$$dX = (AX + F(X))dt + G(X)dL, \quad (4.1)$$

with one dimensional Lévy process L and general transformations F, G acting on the Hilbert state space H . They are some improvements of the classical results. Their proofs are given at the end of the present section.

Theorem 4.1 *Assume that*

$$\|F(x)\|_H + \|G(x)\|_H \leq c(1 + |x|)$$

for some $c > 0$ and for each $R > 0$ there exists $c_R > 0$ such that for all $x, y \in H$ satisfying $|x| \leq R, |y| \leq R$,

$$\|F(x) - F(y)\| + \|G(x) - G(y)\|_H \leq c_R |x - y|.$$

Then there exists a unique càdlàg weak solution to the equation (4.1).

The following theorem is an extension of a result of Milian [11] to the equations with locally Lipschitz coefficients.

Theorem 4.2 Assume that the equation (4.1), with a Wiener process L , admits a solution X . Assume, in addition, that A generates a strongly continuous semigroup $S_t, t \geq 0$ in $H = L^2(E, \mu)$, with μ being a σ -finite measure on E , and that the semigroup preserves positivity. Assume that for each R there exists a constant C_R such that

$$\|F(x) - F(y)\|_H + \|G(x) - G(y)\|_H \leq C_R \|x - y\|_H, \quad x, y \in B_R, \quad (4.2)$$

where $B_R := \{z \in H; \|z\|_H \leq R\}$. If for each $f \in H_+ \cap C_c^\infty(E)$ and $\varphi \in H_+ \cap C(E)$ such that $\langle \varphi, f \rangle = 0$ the following holds

$$\langle F(\varphi), f \rangle \geq 0 \quad (4.3)$$

$$\langle G(\varphi), f \rangle = 0, \quad (4.4)$$

then $X \geq 0$. Conversely, if all solutions to (4.1), starting from non-negative initial conditions, stay non-negative, then (4.3) and (4.4) hold.

4.1 Proof of Theorem 3.1

We will use Theorem 4.2 in a similar way as in [14]. Let us consider the Lévy-Itô decomposition of L

$$L(t) = at + qW(t) + L_0(t) + L_1(t), \text{ where} \\ L_0(t) := \int_0^t \int_{|y| \leq 1} y \hat{\pi}(ds, dy), \quad L_1(t) := \int_0^t \int_{|y| > 1} y \pi(ds, dy),$$

and a sequence of its approximations of the form

$$L^n(t) = at + qW(t) - tm_n + (L_0^n(t) + L_1(t)),$$

with $L_0^n(t) := \int_0^t \int_{\{\frac{1}{n} < |y| \leq 1\}} y \pi(dy)$ and $m_n := E[L_0^n(1)]$. Here π stands for the random Poisson measure of L and $\hat{\pi}$ for its compensated measure.

The equation (2.7) preserves positivity if and only if for each n the equation

$$dr_n(t, x) = \left(\frac{d}{dx} r_n(t, x) + \left(J' \left(\int_0^x g(y, r_n(t, y)) dy \right) + a - m_n \right) g(x, r_n(t, x)) \right) dt \\ + g(x, r_n(t-, x)) (dL_0^n(t) + dL_1(t) + qdW(t)), \quad (4.5)$$

does. As the sum $L_0^n(t) + L_1(t)$ is a compound Poisson process with jumps greater than $\frac{1}{n}$, the driving noise in (4.5) between the jumps is the Wiener process only. Thus we may use the result of Milian. The conditions

$$\int_0^{+\infty} \left(J' \left(\int_0^x g(v, \varphi(v)) dv \right) + a - m_n \right) g(x, \varphi(x)) f(x) e^{\gamma x} dx \geq 0, \\ \int_0^{+\infty} g(x, \varphi(x)) f(x) e^{\gamma x} dx = 0,$$

are satisfied for any $\varphi, f \in L^{2,\gamma}$ such that $\langle \varphi, f \rangle = 0$ if and only if $g(x, 0) = 0$. The solution remains positive in the moment of jump of L^n if and only if

$$r + g(x, r)u \geq 0, \quad r \geq 0, u \in \text{supp}\{\nu\} \cup \left[\frac{1}{n}, +\infty \right)$$

Passing to the limit $n \rightarrow +\infty$ we obtain (3.1). □

4.2 Proofs of Theorem 3.2, Theorem 3.7 and Theorem 3.4, Theorem 3.9

For the proofs of the existence results from Section 3 it is enough to establish local Lipschitz property and linear growth for F, G in $L^{2,\gamma}$ and $H^{1,\gamma}$ respectively, formulated as Proposition 4.3 and Proposition 4.5. Then Theorem 3.2 and Theorem 3.4 follow from Theorem 3.1 and the fact that locally Lipschitz coefficients imply existence of local solutions, see [14]. Theorem 3.7 and Theorem 3.9 can be deduced from Theorem 4.1 and Theorem 3.1.

4.2.1 Local Lipschitzianity and linear growth of the coefficients in $L^{2,\gamma}$

As in the space $L^{2,\gamma}$ the Lipschitz condition of g implies linear growth and Lipschitz property of G , below we formulate the results for F only.

Proposition 4.3 *Assume that (G1) is satisfied.*

a) *If J' is bounded on $[0, +\infty)$ then F has linear growth.*

b) *If J' is locally Lipschitz then F is locally Lipschitz.*

Proof of Proposition 4.3: Let $C_1 := \sup_{z \geq 0} J'(z) < +\infty$.

a) The following estimations hold

$$\begin{aligned} \|F(r)\|_{L^{2,\gamma}} &= \int_0^{+\infty} \left[J' \left(\int_0^x g(y, r(y)) dy \right) g(x, r(x)) \right]^2 e^{\gamma x} dx \leq C_1 \int_0^{+\infty} [g(x, r(x))]^2 e^{\gamma x} dx \\ &\leq C_1 \int_0^{+\infty} [g(x, r(x)) - g(x, 0)]^2 e^{\gamma x} dx \leq C_1 C^2 \int_0^{+\infty} r^2(x) e^{\gamma x} dx \leq C_1 C^2 \|r\|_{L^{2,\gamma}}^2. \end{aligned}$$

b) For any $r, \bar{r} \in L^{2,\gamma}$ we have

$$\begin{aligned} \|F(r) - F(\bar{r})\|_{L^{2,\gamma}}^2 &= \int_0^{+\infty} \left[J' \left(\int_0^x g(y, r(y)) dy \right) g(x, r(x)) - J' \left(\int_0^x g(y, \bar{r}(y)) dy \right) g(x, \bar{r}(x)) \right]^2 e^{\gamma x} dx \\ &\leq 2I_1 + 2I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \int_0^{+\infty} \left[J' \left(\int_0^x g(y, r(y)) dy \right) - J' \left(\int_0^x g(y, \bar{r}(y)) dy \right) \right]^2 g^2(x, r(x)) e^{\gamma x} dx, \\ I_2 &:= \int_0^{+\infty} \left[J' \left(\int_0^x g(y, \bar{r}(y)) dy \right) \right]^2 \left(g(x, \bar{r}(x)) - g(y, r(x)) \right)^2 e^{\gamma x} dx. \end{aligned}$$

Let us notice that in view of (10.4) in Appendix we have

$$\int_0^x g(y, r(y)) dy = \int_0^x (g(y, r(y)) - g(y, 0)) dy \leq C \int_0^x r(y) dy \leq \frac{C}{\sqrt{\gamma}} \|r\|_{L^{2,\gamma}}.$$

Denoting by $D = D(\|r\|_{L^{2,\gamma}}, \|\bar{r}\|_{L^{2,\gamma}})$ the local Lipschitz constant of J' we thus obtain

$$\begin{aligned}
I_1 &\leq D \int_0^{+\infty} \left[\int_0^x (g(y, r(y)) - g(y, \bar{r}(y))) dy \right]^2 g^2(x, r(x)) e^{\gamma x} dx \\
&\leq D \|g(\cdot, r(\cdot)) - g(\cdot, \bar{r}(\cdot))\|_{L^{2,\gamma}}^2 \int_0^{+\infty} g^2(x, r(x)) e^{\gamma x} dx \\
&\leq D \|g(\cdot, r(\cdot)) - g(\cdot, \bar{r}(\cdot))\|_{L^{2,\gamma}}^2 \cdot \int_0^{+\infty} (g(x, r(x)) - g(x, 0))^2 e^{\gamma x} dx \\
&\leq DC^2 \int_0^{+\infty} (r(x) - \bar{r}(x))^2 e^{\gamma x} dx \cdot C^2 \int_0^{+\infty} r^2(x) e^{\gamma x} dx \\
&\leq DC^4 \|r - \bar{r}\|_{L^{2,\gamma}}^2 \|r\|_{L^{2,\gamma}}^2.
\end{aligned}$$

Similarly, for a local boundary B of J' we get

$$I_2 \leq BC^2 \|r - \bar{r}\|_{L^{2,\gamma}}^2,$$

and thus local Lipschitz property follows. \square

4.2.2 Local Lipschitzianity and linear growth of the coefficients in $H^{1,\gamma}$

Let us start from the auxiliary result.

Lemma 4.4 *If $r \in H^{1,\gamma}$ then*

$$\sup_{x \geq 0} |r(x)| \leq 2 \left(\frac{1}{\gamma} \right)^{1/2} \|r\|_{H^{1,\gamma}}.$$

Proof of Lemma 4.4: Integrating by parts gives

$$\int_0^x y \frac{dr(y)}{dy} dy = yr(y)|_0^x - \int_0^x r(y) dy,$$

and thus

$$\begin{aligned}
|xr(x)| &\leq \left| \int_0^x y \frac{dr(y)}{dy} dy \right| + \left| \int_0^x r(y) dy \right| \\
&\leq \left(\int_0^{+\infty} e^{-\gamma y} y^2 dy \right)^{1/2} \left(\int_0^{+\infty} e^{\gamma y} \left(\frac{dr(y)}{dy} \right)^2 dy \right)^{1/2} + \left(\int_0^{+\infty} e^{-\gamma y} dy \right)^{1/2} \left(\int_0^{+\infty} e^{\gamma y} r^2(y) dy \right)^{1/2} \\
&\leq \left(\frac{2}{\gamma^3} \right)^{1/2} \|r\|_{H^{1,\gamma}} + \left(\frac{1}{\gamma} \right)^{1/2} \|r\|_{L^{2,\gamma}}.
\end{aligned}$$

In particular

$$\lim_{x \rightarrow +\infty} r(x) = 0.$$

Moreover,

$$\begin{aligned}
|r(x) - r(0)| &= \left| \int_0^x \frac{dr}{dy}(y) dy \right| \leq \int_0^x e^{-\gamma y/2} e^{\gamma y/2} \left| \frac{dr}{dy}(y) \right| dy \\
&\leq \left(\int_0^{+\infty} e^{-\gamma y} dy \right)^{1/2} \left(\int_0^{+\infty} e^{\gamma y} \left(\frac{dr}{dy}(y) \right)^2 dy \right)^{1/2} \leq \left(\frac{1}{\gamma} \right)^{1/2} \|r\|_{H^{1,\gamma}}.
\end{aligned}$$

Consequently,

$$|r(0)| \leq \left(\frac{1}{\gamma}\right)^{1/2} \|r\|_{H^{1,\gamma}},$$

and therefore

$$\sup_{x \geq 0} |r(x)| \leq 2 \left(\frac{1}{\gamma}\right)^{1/2} \|r\|_{H^{1,\gamma}}.$$

□

Proposition 4.5 *Assume that (G1) is satisfied.*

a) *If J' and J'' are bounded on $[0, +\infty)$ and (G3) holds then G and F have linear growth.*

b) *If J' and J'' are locally Lipschitz and (G2) holds then F and G are locally Lipschitz.*

Proof of Proposition 4.5: (a) Linear growth of G follows from the estimation

$$\begin{aligned} \int_0^{+\infty} \left| \frac{d}{dx} g(x, r(x)) \right|^2 e^{\gamma x} dx &= \int_0^{+\infty} [g'_x(x, r(x)) + g'_y(x, r(x))r'(x)]^2 e^{\gamma x} dx \\ &\leq 2 \int_0^{+\infty} [h(x)]^2 e^{\gamma x} dx + 2 \sup_{x, r \geq 0} |g'_y(x, r)|^2 \int_0^{+\infty} |r'(x)|^2 e^{\gamma x} dx \\ &\leq 2 \|h\|_{L^{2,\gamma}}^2 + 2 \sup_{x, r \geq 0} |g'_y(x, r)|^2 \cdot \|r\|_{H^{1,\gamma}}^2. \end{aligned} \quad (4.6)$$

To show linear growth of F let us start with the inequality

$$\begin{aligned} \int_0^{+\infty} \frac{d}{dx} \left(J' \left(\int_0^x g(v, r(v)) dv \right) g(x, r(x)) \right) &\leq 2 \int_0^{+\infty} \left| J'' \left(\int_0^x g(v, r(v)) dv \right) g^2(x, r(x)) \right|^2 e^{\gamma x} dx \\ &+ \int_0^{+\infty} \left| J' \left(\int_0^{+\infty} g(v, r(v)) dv \right) [g'_x(x, r(x)) + g'_y(x, r(x))r'(x)] \right|^2 e^{\gamma x} dx. \end{aligned}$$

The second integral can be estimated in the same way as (4.6). Linear growth of the first integral follows from the inequality

$$\int_0^{+\infty} |g(x, r(x))|^4 e^{\gamma x} dx \leq \sup_{x, r \geq 0} \left| \frac{g(x, r)}{\sqrt{r}} \right|^4 \int_0^{+\infty} |r(x)|^2 e^{\gamma x} dx.$$

(b) To get the required estimation for G we need to estimate

$$I_0 := \int_0^{+\infty} [g'_y(x, r(x))r'(x) - g'_y(x, \bar{r}(x))\bar{r}'(x)]^2 e^{\gamma x} dx.$$

Using Lemma 4.4 we obtain the following inequalities

$$\begin{aligned} I_0 &\leq 2 \int_0^{+\infty} |g'_y(x, r(x))|^2 |r'(x) - \bar{r}'(x)|^2 e^{\gamma x} dx + \int_0^{+\infty} |g'_y(x, r(x)) - g'_y(x, \bar{r}(x))|^2 |r'(x)|^2 e^{\gamma x} dx \\ &\leq 2 \sup_{x, r \geq 0} |g'_y(x, r)|^2 \int_0^{+\infty} |r'(x) - \bar{r}'(x)|^2 e^{\gamma x} dx \\ &+ 2 \left(\sup_{x, u, v \geq 0} \frac{|g'_y(x, u) - g'_y(x, v)|}{|u - v|} \right)^2 \int_0^{+\infty} |r(x) - \bar{r}(x)|^2 (\bar{r}'(x))^2 e^{\gamma x} dx \\ &\leq 2 \sup_{x, r \geq 0} |g'_y(x, r)|^2 \cdot \|r - \bar{r}\|_{H^{1,\gamma}}^2 \\ &+ 2 \left(\sup_{x, u, v \geq 0} \frac{|g'_y(x, u) - g'_y(x, v)|}{|u - v|} \right)^2 \frac{4}{\gamma} \|r - \bar{r}\|_{H^{1,\gamma}}^2 \cdot \|\bar{r}\|_{H^{1,\gamma}}^2, \end{aligned}$$

and thus local Lipschitz property for G follows.

To show the same for F it is sufficient to show the Lipschitz estimation for the formula

$$I := \int_0^{+\infty} \left[\frac{d}{dx} \left\{ \left(J' \left(\int_0^x g(y, r(y)) dy \right) g(x, r(x)) - J' \left(\int_0^x g(y, \bar{r}(y)) dy \right) g(x, \bar{r}(x)) \right) \right\} \right]^2 e^{\gamma x} dx.$$

By explicit calculations we obtain

$$I \leq 3I_1 + 3I_2 + 3I_3,$$

where

$$I_1 := \int_0^{+\infty} \left[J'' \left(\int_0^x g(y, r(y)) dy \right) g^2(x, r(x)) - J'' \left(\int_0^x g(y, \bar{r}(y)) dy \right) g^2(x, \bar{r}(x)) \right]^2 e^{\gamma x} dx,$$

$$I_2 := \int_0^{+\infty} \left[J' \left(\int_0^x g(y, r(y)) dy \right) g'_x(x, r(x)) - J' \left(\int_0^x g(y, \bar{r}(y)) dy \right) g'_x(x, \bar{r}(x)) \right]^2 e^{\gamma x} dx,$$

$$I_3 := \int_0^{+\infty} \left[J' \left(\int_0^x g(y, r(y)) dy \right) g'_y(x, r(x)) \cdot r'(x) - J' \left(\int_0^x g(y, \bar{r}(y)) dy \right) g'_y(x, \bar{r}(x)) \cdot \bar{r}'(x) \right]^2 e^{\gamma x} dx.$$

We can estimate I_1 in the same way as in the proof of Proposition 4.3. With the use of (i) and (ii) one obtains the estimate for I_2 . I_3 can also be estimated in the same way as in $L^{2,\gamma}$ provided that we have additional inequalities for

$$\int_0^{+\infty} [g'_y(x, r(x))r'(x) - g'_y(x, \bar{r}(x))\bar{r}'(x)]^2 e^{\gamma x} dx$$

which is exactly I_0 and is estimated above, and

$$I_4 := \int_0^{+\infty} [g'_y(x, \bar{r}(x))\bar{r}'(x)]^2 e^{\gamma x} dx.$$

Estimation for I_4 follows from the bound on g'_y . □

4.3 Proof of Theorem 4.1

Let F_n, G_n , $n = 1, 2, \dots$ be such that

(i) $F_n(x) = F(x)$ and $G_n(x) = G(x)$ if $|x| \leq n$,

(ii) for all $t > 0$ and $x \in H$,

$$\|F_n(x)\| + \|G_n(x)\| \leq c(1 + |x|),$$

(iii) there is a constant c_n such that for all $x, y \in H$,

$$\|F_n(x) - F_n(y)\| + \|(G_n(x) - G_n(y))\| \leq c_n|x - y|.$$

By Theorem 9.29 of [14], the equation obtained from (4.1) by replacing F and G by F_n and G_n , has a unique càdlàg solution X_n starting from any $x_0 \in H$ and satisfying the estimation

$$\sup_{t \leq T} \mathbf{E} \|X_n(t)\|^2 \leq C (1 + \|x_0\|^2), \quad (4.7)$$

for some $C > 0$.

Let

$$\tau_n := \inf\{t \leq T : \|X_n(t)\| > n\}.$$

On the time interval $[0, \tau_n)$, the trajectories of X_n are contained in the ball $B(0, n)$ in H with center at 0 and radius n , and therefore X_n satisfies (4.1). In particular, for all $m > n$, X_m and X_n coincide on $[0, \tau_n)$. Define $X(t) = X_n(t)$ if $t < \tau_n$. Note that X is well defined. To finish the proof it is enough to show that

$$\lim_{n \rightarrow \infty} P\left(\sup_{s \leq t} \|X_n(s)\| > n\right) = 0.$$

Let n be such that $\|X(0)\| \leq n/3$ for $t \leq T$. Then

$$\begin{aligned} P\left(\sup_{t \leq T} \|X_n(t)\| > n\right) &\leq P\left(\sup_{t \leq T} \left\|\int_0^t S(t-s)F_n(X_n(s))ds\right\| > \frac{n}{3}\right) \\ &\quad + P\left(\sup_{t \leq T} \left\|\int_0^t S(t-s)G_n(X_n(s-))dM(s)\right\| > \frac{n}{3}\right) := I_1 + I_2. \end{aligned}$$

However, for a constant \hat{c} independent of n ,

$$\sup_{t \leq T} \left\|\int_0^t S(t-s)F_n(X_n(s))ds\right\| \leq \hat{c} \left(1 + \int_0^T \|X_n(s)\|ds\right),$$

and hence, by Chebyshev's inequality and (4.7), there is a constant $\hat{\hat{c}}$ such that

$$\begin{aligned} I_1 &\leq \frac{3\hat{c}}{n} \left(1 + \int_0^T \mathbf{E}\|X_n(s)\|ds\right) \\ &\leq \frac{3\hat{c}}{n} \left(1 + \int_0^T (\mathbf{E}\|X_n(s)\|^2)^{1/2} ds\right) \\ &\leq \frac{3\hat{\hat{c}}}{n} (1 + \|x_0\|^2)^{1/2}. \end{aligned}$$

Hence $I_1 \rightarrow 0$ as $n \rightarrow \infty$. By Kotelenetz's inequality (see e.g. [14]) and (4.7) there is a constant \tilde{c} such that

$$\begin{aligned} I_2 &\leq \left(\frac{3}{n}\right)^2 \tilde{c} \mathbf{E} \int_0^T \|G_n(X_n(s))\|^2 ds \\ &\leq 2c \left(\frac{3}{n}\right)^2 \tilde{c} \int_0^T (1 + \mathbf{E}\|X_n(s)\|^2) ds \\ &\leq \tilde{\tilde{c}} \left(\frac{3}{n}\right)^2 (1 + \|x_0\|^2). \end{aligned}$$

Hence $I_2 \rightarrow 0$ as $n \rightarrow \infty$ and the assertion follows. \square

4.4 Proof of Theorem 4.2

Proof of Theorem 4.2: We use the original result of Milian [11]. Let us consider a sequence of transformations

$$F^n(x) := F(x)h^n(\|x\|); \quad G^n(x) := F(x)h^n(\|x\|),$$

where

$$h^n(z) = \begin{cases} 1 & \text{for } z \in [0, n), \\ 2 - \frac{z}{n} & \text{for } z \in [n, 2n), \\ 0 & \text{for } z \geq 2n. \end{cases}$$

One can check that h^n is Lipschitz for each n and thus F^n, G^n are Lipschitz on H . The following hold

$$\langle F^n(\varphi), f \rangle = \begin{cases} \langle F(\varphi), f \rangle \geq 0 & \text{if } \|\varphi\| < n, \\ h^n(\|\varphi\|) \langle F(\varphi), f \rangle \geq 0 & \text{if } n \leq \|\varphi\| < 2n, \\ 0 & \text{for } \|\varphi\| \geq 2n, \end{cases}$$

and $\langle G^n(\varphi), f \rangle = 0$. Therefore it follows that the solution X^n of the equation (4.1) with F, G replaced by F^n, G^n is non-negative. But

$$X^n = X \mathbf{1}_{B_n}(\|X\|),$$

which implies that X is non-negative on each ball. Passing to the limit with the radius and using the fact that X is bounded we obtain positivity of X . Using the arguments in the opposite direction we get necessity of (4.3), (4.4). \square

Part II

HJMM equation with linear diffusion

In this part we assume that

$$g(x, y) = \lambda(x)y, \quad x, y \geq 0$$

where $\lambda(\cdot)$ is a continuous function. Then the weak version of (1.5) is of the form:

$$\begin{aligned} r(t, x) = S_t(r_0)(x) &+ \int_0^t S_{t-s} \left(J' \left(\int_0^x \lambda(v)r(s, v)dv \right) \lambda(x)r(s, x) \right) ds \\ &+ \int_0^t S_{t-s} \left(\lambda(x)r(s-, x) \right) dL(s), \quad x \geq 0, \quad t \in (0, T^*]. \end{aligned} \quad (4.8)$$

The following two conditions (B3) and (B4), already introduced in the Preliminaries, play an essential role in the analysis of existence of the global solutions to the equation (4.8). Roughly speaking solutions explode if,

$$(B3) \quad \text{For some } a > 0, \quad b \in \mathbb{R}, \quad J'(z) \geq a(\ln z)^3 + b, \quad \text{for all } z > 0,$$

and global solutions exist if,

$$(B4) \quad \limsup_{z \rightarrow \infty} (\ln z - \bar{\lambda} T^* J'(z)) = +\infty, \quad 0 < T^* < +\infty.$$

Results on local existence are formulated as Theorem 5.1 and Theorem 5.2 and follow from the general results of the first part. Theorem 5.3 formulates conditions for non-existence of global solutions and is inspired by a similar result in [1]. Subsequent results concern global solutions, see Theorem 5.5, strong solutions, see Theorem 5.7 as well as their uniqueness, see Theorem 5.8.

Some existence results on global solution to (4.8) can be deduced from results of Part I like Theorem 3.7 or Theorem 3.9 however under very restrictive conditions on J' . In fact, we have the following elementary observation.

Proposition 4.6 *If the drift transformation F defined by (1.2) is of linear growth in $L^{2,\gamma}$, then J' is bounded on $[0, +\infty)$. In particular*

$$q = 0, \quad \text{supp}\{\nu\} \subseteq [0, +\infty) \quad \text{and} \quad \int_0^{+\infty} y\nu(dy) < +\infty.$$

Proof: Assume, to the contrary, that J' is unbounded and define

$$r_n(x) = n\mathbf{1}_{[1,3]}(x), \quad n = 1, 2, \dots$$

As for sufficiently large $z \geq 0$ the function $(J'(z))^2$ is increasing we have, for large n

$$\frac{\|F(r_n)\|^2}{\|r_n\|^2} = \frac{\int_1^3 \left(J' \left(\int_1^x \lambda(y)ndy \right) \right)^2 \lambda^2(x)n^2e^{\gamma x}dx}{n^2 \int_1^3 e^{\gamma x}dx} \geq \frac{\lambda^2 \int_1^3 \left(J' \left(\lambda n(x-1) \right) \right)^2 e^{\gamma x}dx}{\int_1^3 e^{\gamma x}dx}.$$

Since,

$$\int_1^3 \left(J'(\underline{\lambda}n(x-1)) \right)^2 e^{\gamma x} dx \geq \int_2^3 \left(J'(\underline{\lambda}n(x-1)) \right)^2 e^{\gamma x} dx \geq \left(J'(\underline{\lambda}n) \right)^2 \int_2^3 e^{\gamma x} dx \xrightarrow{n} +\infty,$$

the main claim holds. The rest follows from Proposition 2.2. \square

5 Formulation of the main results

EXISTENCE OF LOCAL SOLUTIONS

The following theorem is a direct consequence of Theorem 3.2.

Theorem 5.1 *Assume that:*

$$(\Lambda 0) \quad \lambda \text{ is continuous and } \inf_{x \geq 0} \lambda(x) = \underline{\lambda} > 0, \quad \sup_{x \geq 0} \lambda(x) = \bar{\lambda} < +\infty,$$

$$(\Lambda 1) \quad \text{supp } \nu \subseteq \left[-\frac{1}{\lambda}, +\infty\right)$$

$$(L1) \quad \int_1^{+\infty} y^2 \nu(dy) < +\infty,$$

hold. Then there exists a unique local weak solution to the equation (4.8) taking values in the space $L_+^{2,\gamma}$.

In the formulation of the theorem a simplified, but under $(\Lambda 0), (\Lambda 1)$, equivalent version of the condition $(L1)$ from the Preliminaries was used. In fact the positivity assumptions $(G1)(i)$, $(G1)(ii)$ follow from $(\Lambda 0), (\Lambda 1)$ and the assumption $(G1)(iii)$ follows from $(\Lambda 0)$. Local Lipschitzianity is a consequence of $(L1)$, see Proposition 2.1 and Proposition 4.3.

Similarly as a consequence of Theorem 3.4 we obtain the following local existence result in $H_+^{1,\gamma}$.

Theorem 5.2 *Assume that conditions $(\Lambda 0), (\Lambda 1)$,*

$$(\Lambda 2) \quad \lambda, \lambda' \text{ are bounded and continuous on } \mathbb{R}_+,$$

and

$$(L2) \quad \int_1^{+\infty} y^3 \nu(dy) < +\infty,$$

are satisfied. Then there exists a unique local weak solution to the equation (4.8) taking values in the space $H_+^{1,\gamma}$.

NON-EXISTENCE OF GLOBAL SOLUTIONS IN $H_+^{1,\gamma}$

Our first result on global solutions is of negative type.

Theorem 5.3 *Assume that conditions $(\Lambda 0), (\Lambda 1)$,*

$$(\Lambda 3) \quad \lambda, \lambda', \lambda'', \text{ are bounded and continuous on } \mathbb{R}_+,$$

$$(B0) \quad \int_1^{+\infty} y\nu(dy) < +\infty,$$

$$(B3) \quad J'(z) \geq a(\ln z)^3 + b, \quad \text{for some } a > 0, b \in \mathbb{R}, \text{ and all } z > 0$$

are satisfied.

Then, for some $k > 0$ and all $r_0(\cdot) \in H_+^{1,\gamma}$ such that $r_0(x) \geq k, \forall x \in [0, T^*]$, the global solution in $H_+^{1,\gamma}$ of (4.8) does not exist on the interval $[0, T^*]$.

It follows from Theorem 5.2 and Theorem 5.3 that if conditions $(\Lambda 0), (\Lambda 1), (\Lambda 2), (\Lambda 3), (B0), (B3)$ and $(L2)$ hold then any local solution in $H_+^{1,\gamma}$ explodes.

The theorem remains true if the condition $(B3)$ is replaced by a stronger but a more explicit condition on the measure ν , see Proposition 2.4 .

Theorem 5.4 Assume that conditions $(\Lambda 0), (\Lambda 1)$,

$$(\Lambda 3) \quad \lambda, \lambda', \lambda'', \text{ are bounded and continuous on } \mathbb{R}_+,$$

$$(B0) \quad \int_1^{+\infty} y\nu(dy) < +\infty,$$

$$(B5) \quad \int_0^x y^2\nu(dy) \sim x^\rho \cdot M(x), \quad \text{as } x \rightarrow 0,$$

where M is a slowly varying function, at 0 and $\rho < 1$, are satisfied.

Then, for some $k > 0$ and all $r_0(\cdot) \in H_+^{1,\gamma}$ such that $r_0(x) \geq k, \forall x \in [0, T^*]$, the global solution in $H_+^{1,\gamma}$ of (4.8) does not exist on the interval $[0, T^*]$.

EXISTENCE OF GLOBAL SOLUTIONS

We have the following existence result in which the key role is played by the logarithmic growth condition $(B4)$. Condition $(B2)$, which appears in its formulation, was introduced in Proposition 2.2.

Theorem 5.5 Assume that $(\Lambda 0), (\Lambda 1)$ and conditions

$$(\Lambda 2) \quad \lambda, \lambda' \text{ are bounded and continuous on } \mathbb{R}_+,$$

$$(B0) \quad \int_1^{+\infty} y\nu(dy) < +\infty,$$

$$(B4) \quad \limsup_{z \rightarrow \infty} (\ln z - \bar{\lambda} T^* J'(z)) = +\infty, \quad 0 < T^* < +\infty, \text{ hold.}$$

(a) If $r_0 \in L_+^{2,\gamma}$ then there exists a solution to (4.8) taking values in the space $L_+^{2,\gamma}$.

(b) Assume, in addition, that

$$(\Lambda 3) \quad \lambda, \lambda', \lambda'', \text{ are bounded and continuous on } \mathbb{R}_+,$$

$$(B2) \quad \text{supp}\{\nu\} \subseteq [0, +\infty) \quad \text{and} \quad \int_1^\infty y^2\nu(dy) < \infty.$$

If $r_0 \in H_+^{1,\gamma}$ then there exists a solution to (4.8) taking values in the space $H_+^{1,\gamma}$.

The condition (B4) implies, like in the general diffusion case, that the process L should be without Wiener part and without negative jumps. On the other hand Theorem 5.3 shows that under (B3) the absence of the Wiener part and of the negative jumps in the decomposition of L is also necessary for existence in the linear case in $H_+^{1,\gamma}$.

The theorem remains true if the condition (B4) is replaced by a stronger but a specific condition on the measure ν , see Proposition 2.4 .

Theorem 5.6 *Assume that $(\Lambda 0), (\Lambda 1)$ and conditions*

$$(A2) \quad \lambda, \lambda' \text{ are bounded and continuous on } \mathbb{R}_+,$$

$$(B0) \quad \int_1^{+\infty} y \nu(dy) < +\infty,$$

$$(B5) \quad \int_0^x y^2 \nu(dy) \sim x^\rho \cdot M(x), \quad \text{as } x \rightarrow 0,$$

where M is a slowly varying function, at 0 and $\rho > 1$, are satisfied.

(a) If $r_0 \in L_+^{2,\gamma}$ then there exists a solution to (4.8) taking values in the space $L_+^{2,\gamma}$.

(b) Assume, in addition, that

$$(\Lambda 3) \quad \lambda, \lambda', \lambda'', \text{ are bounded and continuous on } \mathbb{R}_+,$$

$$(B2) \quad \text{supp}\{\nu\} \subseteq [0, +\infty) \quad \text{and} \quad \int_1^\infty y^2 \nu(dy) < \infty.$$

If $r_0 \in H_+^{1,\gamma}$ then there exists a solution to (4.8) taking values in the space $H_+^{1,\gamma}$.

Our assumptions implying global existence are not very restrictive. The condition (B4) is weaker than the requirement that J' is bounded, which was necessary for the standard contraction principle approach, see Proposition 4.6. Moreover, the assumptions do not imply local Lipschitz property of the coefficients. In Theorem 5.5 we need (B4) and integrability of ν outside of the unit ball, that is

$$\int_1^{+\infty} y \nu(dy) < +\infty, \tag{5.1}$$

for the space $L_+^{2,\gamma}$ and

$$\text{supp}\{\nu\} \subseteq [0, +\infty) \quad \text{and} \quad \int_1^{+\infty} y^2 \nu(dy) < +\infty, \tag{5.2}$$

for $H_+^{1,\gamma}$. It is clear that

$$(5.1) \quad \Leftrightarrow \quad \int_1^{+\infty} y^2 \nu(dy) < +\infty, \tag{5.3}$$

and

$$(5.2) \quad \Leftrightarrow \quad \int_1^{+\infty} y^3 \nu(dy) < +\infty. \tag{5.4}$$

Under the condition $\text{supp}\{\nu\} \subseteq [-\frac{1}{\lambda}, +\infty)$, the right hand sides of (5.3), (5.4) are equivalent to conditions (L1), (L2) which in turn correspond to local Lipschitz properties of F and G in $L^{2,\gamma}$,

resp. $H^{1,\gamma}$. On the other hand, as explained in the Preliminaries, (B4) is related to the behavior of ν close to zero. Thus for each Lévy process satisfying (B4) and (5.1) (or (5.2)) there exists a global solution in $L_+^{2,\gamma}$, (resp. $H_+^{1,\gamma}$) but F, G are not locally Lipschitz.

EXISTENCE OF STRONG SOLUTIONS IN $H_+^{1,\gamma}$

Under additional conditions we can establish existence of the strong solutions to (1.5).

Theorem 5.7 *Assume that $\lambda(x) \equiv \lambda$ is constant and all assumptions of Theorem 5.5 (b) are satisfied. Then the weak non-exploding solution given by Theorem 5.5 (b) is a strong solution of (1.5).*

UNIQUENESS OF THE GLOBAL SOLUTION IN $H_+^{1,\gamma}$

Assumptions of Theorem 5.5 do not imply, in general, the uniqueness of the solutions. Also this property does not follow from the uniqueness of the local solutions. Thus the following theorem cannot be deduced from the contraction principle.

Theorem 5.8 *Assume that*

$$(B2) \quad \text{supp}\{\nu\} \subseteq [0, +\infty) \quad \text{and} \quad \int_1^\infty y^2 \nu(dy) < \infty.$$

If there exists a solution of the equation (4.8) on the interval $[0, T^]$ taking values in $H_+^{1,\gamma}$ then the solution is unique.*

EQUIVALENT EQUATION

We pass now to the formulation of an equivalence result indicated in the introduction. It is of independent interest and will serve as the main technical tool in the majority of the proofs.

A random field $r(t, x)$, $t \in [0, T^*]$, $x \geq 0$, is said to be a solution, in $L^{2,\gamma}$, respectively in $H^{1,\gamma}$, to the *integral equation*:

$$r(t, x) = a(t, x) e^{\int_0^t J'(\int_0^{t-s+x} \lambda(v) r(s, v) dv) \lambda(t-s+x) ds}, \quad x \geq 0, \quad t \in [0, T^*], \quad (5.5)$$

where, for $x \geq 0$, $t \in (0, T^*]$,

$$\begin{aligned} a(t, x) := & r_0(t+x) e^{\int_0^t \lambda(t-s+x) dL(s) - \frac{\gamma^2}{2} \int_0^t \lambda^2(t-s+x) ds} \\ & \cdot \prod_{0 \leq s \leq t} (1 + \lambda(t-s+x)(L(s) - L(s-))) e^{-\lambda(t-s+x)(L(s) - L(s-))}, \end{aligned} \quad (5.6)$$

if $r(t, \cdot)$, $t \in [0, T^*]$, is $L^{2,\gamma}$, respectively $H^{1,\gamma}$ valued, bounded and adapted process such that, for each $t \in [0, T^*]$, the equation (5.5) holds for almost all $x > 0$, in the case of $L^{2,\gamma}$, and for all $x \geq 0$, in the case of $H^{1,\gamma}$.

The random field a will be called *the random factor* of the equation (5.5).

Theorem 5.9 *Let r be a solution of (4.8) in the state space $H_+^{1,\gamma}$. Then $r(\cdot, \cdot)$ is a solution of (5.5) in $H_+^{1,\gamma}$.*

Under additional assumptions the converse result is true.

Theorem 5.10 *Assume that conditions $(\Lambda 0)$, $(\Lambda 1)$ and*

$$(B0) \quad \int_1^{+\infty} y\nu(dy) < +\infty,$$

are satisfied.

a) If

$$(\Lambda 2) \quad \lambda, \lambda' \text{ are bounded and continuous on } \mathbb{R}_+,$$

and $r(\cdot)$ is a bounded solution in $L_+^{2,\gamma}$ of (5.5), then $r(\cdot)$ is a càdlàg process in $L_+^{2,\gamma}$ and solves (4.8).

b) If

$$(\Lambda 3) \quad \lambda, \lambda', \lambda'', \text{ are bounded and continuous on } \mathbb{R}_+,$$

$$(B2) \quad \text{supp}\{\nu\} \subseteq [0, +\infty) \quad \text{and} \quad \int_1^\infty y^2\nu(dy) < \infty,$$

and $r(\cdot)$ is a bounded solution in $H_+^{1,\gamma}$ of (5.5), then $r(\cdot)$ is càdlàg in $H_+^{1,\gamma}$ and solves (4.8).

As a consequence, equations (4.8) and (5.5) are equivalent in $H_+^{1,\gamma}$, while each solution of (5.5) in $L_+^{2,\gamma}$ solves also (4.8).

6 Proofs of the equivalence results

The proofs require representation of the solution in a natural and in a moving frame which is discussed in Section 6.1. The proof of Theorem 5.10 is technically rather involved. In particular it requires auxiliary results concerned with the regularity of the random factor a of the equation (5.5).

6.1 Equations in natural and moving frames

We will need a result on a relation between the solution of the equation (4.8) and its version in the natural frame. To this end let us consider two random fields $\{r(t, x), t, x \geq 0\}$, $\{f(t, T), 0 \leq t \leq T < +\infty\}$ such that for each x and each T they admit the following representation

$$r(t, x) = r_0(t + x) + \int_0^t J' \left(\int_0^{t-s+x} \delta(s, v) dv \right) \delta(s, t - s + x) ds + \int_0^t \delta(s, t - s + x) dL(s), \quad (6.1)$$

$$f(t, T) = f_0(T) + \int_0^t J' \left(\int_s^T \sigma(s, v) dv \right) \sigma(s, T) ds + \int_0^t \sigma(s, T) dL(s), \quad (6.2)$$

for some regular fields $\delta(\cdot, \cdot), \sigma(\cdot, \cdot)$ and initial conditions $r_0(\cdot), f_0(\cdot)$. We have the following auxiliary lemma showing the relation between the dynamics of r and f in the case when $f(t, T) = r(t, T - t)$.

Lemma 6.1 a) Let r be a random field given by (6.1). If $f(t, T) := r(t, T - t)$, $0 \leq t \leq T < +\infty$ then f satisfies (6.2) with $\sigma(t, T) := \delta(t, T - t)$.

b) Let f be a random field given by (6.2). If $r(t, x) := f(t, t + x)$, $t, x \geq 0$ then r satisfies (6.1) with $\delta(t, x) := \sigma(t, t + x)$.

Proof: (a) In virtue of (6.1) we have

$$\begin{aligned} f(t, T) &= r(t, T - t) = r_0(T) + \int_0^t J' \left(\int_0^{T-s} \delta(s, v) dv \right) \delta(s, T - s) ds + \int_0^t \delta(s, T - s) dL(s) \\ &= r_0(T) + \int_0^t J' \left(\int_s^T \delta(s, v - s) dv \right) \delta(s, T - s) ds + \int_0^t \delta(s, T - s) dL(s) \\ &= f_0(T) + \int_0^t J' \left(\int_s^T \sigma(s, v) dv \right) \sigma(s, T) ds + \int_0^t \sigma(s, T) dL(s). \end{aligned}$$

To get (b) we can repeat the calculations in the reversed order. \square

6.2 Proof of Theorem 5.9

Let us consider the solution of (4.8) in a natural frame $f(t, T) := r(t, T - t)$, $0 \leq t \leq T < +\infty$. As convergence in $H^{1,\gamma}$ implies uniform convergence on $[0, +\infty)$, see Lemma 4.4, it follows from the càdlàg property of r in $H_+^{1,\gamma}$ that for each $T > 0$ the process $f(\cdot, T)$ is càdlàg on $[0, T]$. As r satisfies (6.1) with $\delta(t, x) := \lambda(t)r(t-, x)$, it follows from Lemma 6.1 that f satisfies

$$\begin{aligned} f(t, T) &= f_0(T) + \int_0^t J' \left(\int_s^T \lambda(v - s) f(s, v) dv \right) \lambda(T - s) f(s, T) ds \\ &\quad + \int_0^t \lambda(T - s) f(s-, T) dL(s). \end{aligned} \tag{6.3}$$

Thus $f(\cdot, T)$ solves the Doléans-Dade equation

$$df(t, T) = f(t-, T) \left[J' \left(\int_t^T \lambda(v - t) f(t, v) dv \right) \lambda(T - t) dt + \lambda(T - t) dL(t) \right],$$

and admits the following representation, see [16],

$$f(t, T) = \hat{a}(t, T) e^{\int_0^t J' \left(\int_s^T \lambda(v - s) f(s, v) dv \right) \lambda(T - s) ds}, \tag{6.4}$$

with

$$\hat{a}(t, T) := f_0(T) e^{\int_0^t \lambda(T - s) dL(s) - \frac{1}{2} q^2 \int_0^t \lambda^2(T - s) ds} \prod_{0 \leq s \leq t} \left(1 + \lambda(T - s) \triangle L(s) \right) e^{-\lambda(T - s) \triangle L(s)},$$

where

$$\triangle L(s) = L(s) - L(s-), \quad s \geq 0.$$

Putting $T = t + x$, $x \geq 0$ into (6.4) and checking that $\hat{a}(t, t + x) = a(t, x)$ one obtains that r satisfies (5.5). \square

6.3 Proof of Theorem 5.10

The proof is divided into two main steps establishing the regularity of the random factor a and then the regularity of the nonlinear part of (5.5).

6.3.1 Step 1. Regularity of the random factor of (5.5)

Here we are dealing with the regularity of the random fields

$$I_1(t, x) := \int_0^t \lambda(t - s + x) dL(s), \quad t \in [0, T^*], \quad x \geq 0, \quad (6.5)$$

$$I_2(t, x) := \prod_{0 \leq s \leq t} (1 + \lambda(t - s + x) \Delta L(s)) e^{-\lambda(t-s+x) \Delta L(s)}, \quad t \in [0, T^*], \quad x \geq 0, \quad (6.6)$$

appearing in (5.6)

Proposition 6.2 *Let I_1 be given by (6.5). Assume that $(\Lambda 0), (\Lambda 1)$ are satisfied.*

- i) If $(\Lambda 2)$ is satisfied then there exists a version of the random field $I_1(t, x)$ which is bounded on $[0, T^*] \times [0, +\infty)$ and for each $x \geq 0$, the stochastic integral $I_1(\cdot, x)$ is a càdlàg process.*
- ii) If $(\Lambda 3)$ is satisfied then the above assertion is true for the random field $\frac{\partial}{\partial x} I_1(t, x)$, $t \in [0, T^*]$, $x \geq 0$.*

Proof: We will show (i). The proof of (ii) is similar. By Proposition 9.16 of [14] the integration by parts formula holds

$$I_1(t, x) = \int_0^t \lambda(t - s + x) dL(s) = \lambda(x) L(t) + \int_0^t \lambda'(t - s + x) L(s) ds, \quad t, x \geq 0. \quad (6.7)$$

The integral on the right hand side of (6.7) is continuous in t as the convolution of two locally bounded functions. Boundedness follows from the assumption $(\Lambda 2)$. \square

Proposition 6.3 *Let I_2 be given by (6.6) and $(\Lambda 0), (\Lambda 1)$ be satisfied.*

- i) Then I_2 is a bounded field on $[0, T^*] \times [0, +\infty)$ and for each $x \geq 0$ the process $I_2(\cdot, x)$ has càdlàg version.*
- ii) If $(\Lambda 2)$ holds then the above assertion is true for the field $\frac{\partial}{\partial x} I_2(t, x)$, $t \in [0, T^*]$, $x \geq 0$.*

Proof: Under $(\Lambda 0), (\Lambda 1)$ we can write I_2 in the form

$$I_2(t, x) = \int_0^t \int_{-\frac{1}{\lambda}}^{+\infty} [\ln(1 + \lambda(t - s + x)y) - \lambda(t - s + x)y] \pi(ds, dy), \quad t \in [0, T^*], \quad x \geq 0,$$

where $\pi(ds, dx)$ stands for the jump measure of the process L . Let us fix two numbers $a \leq 0$ and $b > 0$ such that

$$|\lambda(z)y| \leq \frac{1}{2}, \quad z \geq 0, \quad y \in [a, b]. \quad (6.8)$$

Outside of the set $[0, T^*] \times [a, b]$ the measure π consists of finite numbers of atoms only, so the fields

$$\begin{aligned} & \int_0^t \int_{-\frac{1}{\lambda}}^a [\ln(1 + \lambda(t-s+x)y) - \lambda(t-s+x)y] \pi(ds, dy), \\ & \int_0^t \int_b^{+\infty} [\ln(1 + \lambda(t-s+x)y) - \lambda(t-s+x)y] \pi(ds, dy), \quad t \in [0, T^*], x \geq 0, \end{aligned}$$

are obviously bounded and càdlàg in t . Thus required properties of $I_2(t, x)$ are equivalent to those of the field

$$J(t, x) := \int_0^t \int_a^b [\ln(1 + \lambda(t-s+x)y) - \lambda(t-s+x)y] \pi(ds, dy), \quad t \in [0, T^*], x \geq 0.$$

First we show boundedness. By (6.8) we have

$$| \ln(1 + \lambda(z)y) - \lambda(z)y | \leq \lambda^2(z)y^2, \quad z \geq 0, y \in [a, b],$$

and consequently

$$| J(t, x) | \leq \int_0^t \int_a^b \lambda^2(t-s+x)y^2 \pi(ds, dy), \quad t \in [0, T^*], x \geq 0.$$

Due to $(\Lambda 0), (\Lambda 1)$ boundedness of J follows. Since

$$\begin{aligned} J'_x(t, x) &= \int_0^t \int_a^b \lambda'(t-s+x)y \left[\frac{1}{1 + \lambda(t-s+x)y} - 1 \right] \pi(ds, dx) \\ &= - \int_0^t \int_a^b \frac{\lambda'(t-s+x)\lambda(t-s+x)}{1 + \lambda(t-s+x)y} y^2 \pi(ds, dx), \end{aligned}$$

in view of (6.8), the following estimation holds

$$| J'_x(t, x) | = \int_0^t \int_a^b \frac{| \lambda'(t-s+x)\lambda(t-s+x) |}{\frac{1}{2}} y^2 \pi(ds, dx).$$

Therefore, by $(\Lambda 2)$, boundedness of $J'_x(t, x)$ and thus also $\frac{\partial}{\partial x} I_2(t, x)$ follows.

Below we show càdlàg property for $I_2(\cdot, x)$. The proof for $\frac{\partial}{\partial x} I_2(t, x)$ is the same. We will use the following lemma.

Lemma 6.4 *Assume that $\varphi(t, x, s, y)$, $(t, x) \in [0, T^*] \times [0, +\infty)$, $(s, y) \in [0, T^*] \times [a, b]$, $a < b$, is a continuous and bounded function such that*

$$\varphi(t, x, s, y) = 0 \quad \text{for } s \geq t, x \geq 0, y \in [a, b],$$

and γ is a finite measure on $[0, T^] \times [a, b]$. Then the function*

$$\Phi(t, x) := \int_0^t \int_a^b \varphi(t, x, s, y) \gamma(ds, dy), \quad t \in [0, T^*], x \geq 0,$$

is continuous.

Proof of Lemma 6.4: By the assumptions,

$$\Phi(t, x) := \int_0^{T^*} \int_a^b \varphi(t, x, s, y) \gamma(ds, dy), \quad t \in [0, T^*], \quad x \geq 0.$$

If $(t_n, x_n) \rightarrow (t, x)$ then $\varphi(t_n, x_n, s, y) \rightarrow \varphi(t, x, s, y)$. Since φ is bounded on $[0, T^*] \times [0, +\infty) \times [0, T^*] \times [a, b]$ and γ finite, the result follows from the Lebesgue dominated convergence theorem.

□

Now define a bounded and continuous function

$$j(t, x, s, y) := \frac{1}{y^2} [\ln(1 + \lambda(t - s + x)y) - \lambda(t - s + x)y],$$

for $(t, x) \in [0, T^*] \times [0, +\infty)$, $(s, y) \in [0, T^*] \times [a, b]$. Then

$$J(t, x) = \int_0^t \int_a^b j(t, x, s, y) y^2 \pi(ds, dy).$$

To use Lemma 6.4 let us define

$$\varphi(t, x, s, y) := \begin{cases} j(t, x, s, y) - j(t, x, t, y), & s < t, x \geq 0, y \in [a, b], \\ 0, & s \geq t, x \geq 0, y \in [a, b], \end{cases}$$

and $\gamma(ds, dy) := y^2 \pi(ds, dy)$. Then

$$\begin{aligned} J(t, x) &= \int_0^t \int_a^b \varphi(t, x, s, y) y^2 \pi(ds, dy) \\ &\quad + \int_0^t \int_a^b j(t, x, t, y) y^2 \pi(ds, dy) \\ &= \Phi(t, x) + \int_a^b j(t, x, t, y) y^2 \pi([0, t], dy). \end{aligned}$$

The function Φ is continuous by Lemma 6.4 and thus $J(\cdot, x)$ is càdlàg for any $x \geq 0$. □

We will need one more result concerned with regularity of random fields.

Proposition 6.5 *Let $h = h(x) \in L^{2,\gamma}$ and $H = H(t, x), t \in [0, T^*], x \geq 0$ be a function such that*

$$\sup_{(t,x) \in [0, T^*] \times [0, +\infty)} |H(t, x)| < +\infty,$$

and $H(\cdot, x)$ is càdlàg for each $x \geq 0$. Then the function $\tilde{h} : [0, T^] \rightarrow L^{2,\gamma}$ defined by*

$$\tilde{h} := h(t + x)H(t, x)$$

is càdlàg in $L^{2,\gamma}$.

Proof: We have the following estimation

$$\begin{aligned}
\|\tilde{h}(t) - \tilde{h}(s)\|_{L^{2,\gamma}}^2 &= \int_0^{+\infty} |h(t+x)H(t,x) - h(s+x)H(s,x)|^2 e^{\gamma x} dx \\
&= \int_0^{+\infty} |h(s+x)[H(t,x) - H(s,x)] + [h(t+x) - h(s+x)]H(t,x)|^2 e^{\gamma x} dx \\
&\leq 2e^{-\gamma s} \int_0^{+\infty} e^{\gamma(x+s)} |h(s+x)|^2 |H(t,x) - H(s,x)|^2 dx + 2C\|S_t(h) - S_s(h)\|_{L^{2,\gamma}}^2,
\end{aligned}$$

where $C = \sup_{(t,x) \in [0, T^*] \times [0, +\infty)} |H(t,x)|$. Using the dominated convergence theorem we see that the limit for the first integral when $s \rightarrow t$ exists and is equal to zero when $s \downarrow t$. The second integral disappears when $s \rightarrow t$ because the semigroup is strongly continuous in $L^{2,\gamma}$. Thus \tilde{h} is a càdlàg function in $L^{2,\gamma}$. \square

6.3.2 Step 2. A priori regularity of the solution

Let us write (5.5) in the form

$$r(t, x) = r_0(t+x)B(t, x),$$

where $B(t, x) := b_1(t, x)I_2(t, x)b_2(t, x)$ and

$$\begin{aligned}
b_1(t, x) &:= e^{I_1(t, x) - \frac{q^2}{2} \int_0^t \lambda^2(t-s+x) ds}, \\
b_2(t, x) &:= e^{\int_0^t J'(\int_0^{t-s+x} \lambda(v)r(s, v) dv) \lambda(t-s+x) ds},
\end{aligned}$$

where $I_1(t, x), I_2(t, x)$ are defined in (6.5) and (6.6).

(a) First we will show that r is càdlàg in $L^{2,\gamma}$. We will show that $\sup_{(t,x) \in [0, T^*] \times [0, +\infty)} |B(t, x)| < +\infty$ and $B(\cdot, x)$ is càdlàg for each x . Then the assertion follows from Proposition 6.5.

It follows from Proposition 6.2 and Proposition 6.3 that $(\Lambda 0), (\Lambda 1)$ and $(\Lambda 2)$ imply that $b_1(t, x)$ and $I_2(t, x)$ are bounded and càdlàg in t . It is clear that $b_2(\cdot, x)$ is continuous. By $(\Lambda 1)$ and $(B 0)$ the function J' is well defined on $[0, +\infty)$. In view of (10.4) we have

$$0 \leq \int_0^{t-s+x} \lambda(v)r(s, v) dv \leq \frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_{t \in [0, T^*]} \|r(t)\|_{L^{2,\gamma}}.$$

and thus the inequality

$$\int_0^t J' \left(\int_0^{t-s+x} \lambda(v)r(s, v) dv \right) \lambda(t-s+x) ds \leq \bar{\lambda} T^* J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|r(t)\|_{L^{2,\gamma}} \right) \wedge 0,$$

holds. Thus $b_2(\cdot, \cdot)$ is bounded on $[0, T^*] \times [0, +\infty)$.

Now we will argue that r is a solution of (4.8). Putting $x = T - t$ we see that the solution in the natural frame satisfies

$$f(t, T) = \hat{a}(t, T) e^{\int_0^t J'(\int_s^T \lambda(v-s)f(s, v) dv) \lambda(T-s) ds}, \quad 0 \leq t \leq T \leq T^*, \quad (6.9)$$

where

$$\hat{a}(t, T) := f_0(T) e^{\int_0^t \lambda(T-s) dL(s) - \frac{\gamma^2}{2} \int_0^t \lambda^2(T-s) ds} \cdot \prod_{0 \leq s \leq t} \left(1 + \lambda(T-s) \Delta L(s)\right) e^{-\lambda(T-s) \Delta L(s)}.$$

For each fixed T the process $f(\cdot, T)$ is a stochastic exponential and thus admits the representation

$$f(t, T) = f(0, T) + \int_0^t J' \left(\int_s^T \lambda(v-s) f(s, v) dv \right) \lambda(T-s) f(s, T) ds + \int_0^t \lambda(T-s) f(s-, T) dL(s).$$

Now the assertion follows from Lemma 6.1.

(b) To show that r is càdlàg in $H^{1,\gamma}$ we use the equality

$$\|r(t) - r(s)\|_{H^{1,\gamma}}^2 = \|r(t) - r(s)\|_{L^{2,\gamma}}^2 + \|r'(t) - r'(s)\|_{L^{2,\gamma}}^2.$$

Thus in view of (a) it is enough to show that $r'(t)$ is càdlàg in $L^{2,\gamma}$. Differentiating (5.5) yields

$$r'(t, x) = r'_0(t+x) b(t, x) b_2(t, x) + r_0(t+x) b'(t, x) b_2(t, x) + r_0(t+x) b(t, x) b'_2(t, x)$$

where $b(t, x) = b_1(t, x) I_2(t, x)$. It follows from (a) that $r'_0(t+x) b(t, x) b_2(t, x)$ is càdlàg in $L^{2,\gamma}$. In view of Proposition 6.2 and Proposition 6.3, (Λ2) and (Λ3) imply that $b'(t, x)$ is bounded and càdlàg in t , so $r_0(t+x) b'(t, x) b_2(t, x)$ is càdlàg in $L^{2,\gamma}$. To finish the proof we need to show that b'_2 is bounded and càdlàg in t . We have

$$\begin{aligned} b'_2(t, x) = b_2(t, x) \Big\{ & \int_0^t J'' \left(\int_0^{t-s+x} \lambda(v) r(s, v) dv \right) \lambda^2(t-s+x) r(s, t-s+x) ds \\ & + \int_0^t J' \left(\int_0^{t-s+x} \lambda(v) r(s, v) dv \right) \lambda'(t-s+x) ds \Big\}. \end{aligned}$$

The assumptions (Λ1) and (B2) guarantee that J'' is continuous on $[0, +\infty)$ and thus locally bounded. In view of Lemma 4.4 we obtain

$$\begin{aligned} & \sup_{(t,x) \in [0, T^*] \times [0, +\infty)} \int_0^t J'' \left(\int_0^{t-s+x} \lambda(v) r(s, v) dv \right) \lambda^2(t-s+x) r(s, t-s+x) ds \\ & \leq \bar{\lambda}^2 \sup_{z \in [0, \frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|r(t)\|_{L^{2,\gamma}}]} |J''(z)| \cdot 2T^* \left(\frac{1}{\gamma} \right)^{\frac{1}{2}} \sup_t \|r(t)\|_{H^{1,\gamma}_+}. \end{aligned}$$

By monotonicity of J' and boundedness of λ' one gets

$$\begin{aligned} & \sup_{(t,x) \in [0, T^*] \times [0, +\infty)} \int_0^t J' \left(\int_0^{t-s+x} \lambda(v) r(s, v) dv \right) \lambda'(t-s+x) ds \\ & \leq T^* \sup_{x \geq 0} |\lambda'(x)| \cdot J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|r(t)\|_{L^{2,\gamma}} \right), \end{aligned}$$

and boundedness of b'_2 follows. The proof that r solves (4.8) is the same as in (a). \square

7 Proofs of necessary conditions for existence in $H_+^{1,\gamma}$

Proof of Theorem 5.3: Assume to the contrary that r is a global solution of (4.8) on $[0, T^*]$ in the space $H_+^{1,\gamma}$. In view of Lemma 6.1 the solution in a moving frame $f(t, T) = r(t, T - t)$, $0 \leq t \leq T \leq T^*$ satisfies

$$\begin{aligned} f(t, T) = f_0(T) + \int_0^t J' \left(\int_s^T \lambda(v - s) f(s, v) dv \right) \lambda(T - s) f(s, T) ds \\ + \int_0^t \lambda(T - s) f(s, T) dL(s), \end{aligned} \quad (7.1)$$

which is the equation studied in [1]. Assumptions $(\Lambda 0), (\Lambda 1), (\Lambda 3), (B 0)$ imply the conditions $(A 1) - (A 4)$ in [1].

We check f is as regular as required in [1]. Since r is adapted and càdlàg in $H_+^{1,\gamma}$, it follows that

(a) $f(\cdot, T)$ is adapted and càdlàg for each $T \in [0, T^*]$,

(b) $f(t, \cdot)$ is continuous.

Using Lemma 4.4 and the fact that r is bounded on $[0, T^*]$, as a càdlàg process in $H_+^{1,\gamma}$, we obtain

$$\sup_{t \in [0, T^*], x \geq 0} |r(t, x)| = \sup_{t \in [0, T^*]} \sup_{x \geq 0} |r(t, x)| \leq 2 \left(\frac{1}{\gamma} \right)^{\frac{1}{2}} \sup_{t \in [0, T^*]} \|r\|_{H_+^{1,\gamma}} < +\infty,$$

which clearly implies that

$$(c) \quad \sup_{0 \leq t \leq T \leq T^*} f(t, T) < +\infty.$$

It follows, however, from Theorem 3.4 in [1] that, under (B3), for sufficiently large $k > 0$ there is no solution of (7.1) in the class of random fields satisfying (a) – (c). Hence a contradiction. \square

8 Proofs of existence of global and strong solutions

In view of Theorem 5.10 we can examine equation (5.5) instead directly (4.8). Let us begin with clarifying of the general idea of examining the problem of existence of solution to the equation (5.5). Define the operator \mathcal{K} , acting on functions of two variables, by

$$\mathcal{K}(h)(t, x) = a(t, x) e^{\int_0^t J' \left(\int_0^{t-s+x} \lambda(v) h(s, v) dv \right) \lambda(s, t-s+x) ds}, \quad x \geq 0, \quad t \in [0, T^*], \quad (8.1)$$

where $a(t, x)$ is given by (5.6). Then the equation (5.5) can be compactly written in the form

$$r(t, x) = \mathcal{K}(r)(t, x), \quad t \in [0, T^*], \quad x \geq 0.$$

The problem of existence of solutions will be examined via properties of the iterative sequence of random fields

$$h_0 \equiv 0, \quad h_{n+1} := \mathcal{K}h_n, \quad n = 1, 2, \dots \quad (8.2)$$

Let us write a in the form $a(t, x) = r_0(t + x)b(t, x)$. It follows from Proposition 6.2 and Proposition 6.3 that under (A2) the field b is bounded, i.e.

$$\sup_{t \in [0, T^*], x \geq 0} b(t, x) < \bar{b}, \quad (8.3)$$

where $\bar{b} = \bar{b}(\omega) > 0$. It can be shown by induction that if $r_0 \in L_+^{2, \gamma}$ then $h_n(t)$ is a bounded process in $L_+^{2, \gamma}$ for each n . Indeed assume that for h_n and show for h_{n+1} . In view of (8.3) and the estimate (10.4) in Appendix, we have

$$\begin{aligned} h_{n+1}(t, x) &\leq r_0(t + x) \bar{b} e^{\bar{\lambda} \int_0^t |J'(\int_0^{t-s+x} \lambda(v) h_n(s, v) dv)| ds} \\ &\leq r_0(t + x) \bar{b} e^{\bar{\lambda} T^* \left| J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|h_n(t)\|_{L^{2, \gamma}} \right) \right|}, \end{aligned}$$

and thus $h_{n+1}(t)$ is bounded in $L_+^{2, \gamma}$. It follows from the assumption $\underline{\lambda} > 0$ and the fact that J' is increasing that the sequence $\{h_n\}$ is monotonically increasing and thus there exists $\bar{h} : [0, T^*] \times [0, +\infty) \rightarrow \mathbb{R}_+$ such that

$$\lim_{n \rightarrow +\infty} h_n(t, x) = \bar{h}(t, x), \quad 0 \leq t \leq T^*, x \geq 0. \quad (8.4)$$

Passing to the limit in (8.2), by the monotone convergence, we obtain

$$\bar{h}(t, x) = \mathcal{K}h(t, x), \quad 0 \leq t \leq T^*, x \geq 0.$$

It turns out that properties of the field \bar{h} strictly depend on the growth of the function J' . In the sequel we show that if (B4) holds then $\bar{h}(t)$ is a bounded process in $L_+^{2, \gamma}$, i.e. $\bar{h}(t), t \in [0, T^*]$ is a non-exploding solution of (5.5) in $L_+^{2, \gamma}$. Additional assumptions guarantee that $\bar{h}(t)$ is bounded in $H_+^{1, \gamma}$ and that the solution is unique.

Before presenting the proof we establish an auxiliary result.

Proposition 8.1 *Assume that J' satisfies (B4). If $r_0 \in L_+^{2, \gamma}$ then there exists a positive constant c_1 such that if*

$$\sup_{t \in [0, T^*]} \|h(t)\|_{L_+^{2, \gamma}} \leq c_1$$

then

$$\sup_{t \in [0, T^*]} \|\mathcal{K}h(t)\|_{L_+^{2, \gamma}} \leq c_1.$$

Proof: By (10.4) in Appendix and (8.3), for any $t \in [0, T^*]$, we have

$$\begin{aligned} \|\mathcal{K}h(t, \cdot)\|_{L_+^{2, \gamma}}^2 &= \int_0^{+\infty} |r_0(t + x)b(t, x)|^2 e^{2 \int_0^t J'(\int_0^{t-s+x} \lambda(v) h(s, v) dv) \lambda(t-s+x) ds} e^{\gamma x} dx \\ &\leq \bar{b}^2 \int_0^{+\infty} |r_0(t + x)|^2 e^{2J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|h(t)\|_{L_+^{2, \gamma}} \right) \int_0^t \lambda(t-s+x) ds} e^{\gamma x} dx \\ &\leq \bar{b}^2 \cdot \|r_0\|_{L_+^{2, \gamma}}^2 \cdot \sup_{s \in [0, t], x \geq 0} e^{2J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|h(t)\|_{L_+^{2, \gamma}} \right) \int_0^t \lambda(t-s+x) ds}. \end{aligned}$$

This implies

$$\sup_t \|\mathcal{K}h(t)\|_{L_+^{2,\gamma}} \leq \bar{b} \cdot \|r_0\|_{L_+^{2,\gamma}} \cdot \sup_{t \in [0, T^*], s \in [0, t], x \geq 0} e^{J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \cdot \sup_t \|h(t)\|_{L_+^{2,\gamma}} \right) \int_0^t \lambda(t-s+x) ds},$$

and thus it is enough to find constant c_1 such that

$$\ln \left(\bar{b} \cdot \|r_0\|_{L_+^{2,\gamma}} \right) + \sup_{t \in [0, T^*], s \in [0, t], x \geq 0} J' \left(\frac{\bar{\lambda} c_1}{\sqrt{\gamma}} \right) \int_0^t \lambda(t-s+x) ds \leq \ln c_1. \quad (8.5)$$

If $J'(z) \leq 0$ for each $z \geq 0$ then we put $c_1 = \bar{b} \cdot \|r_0\|_{L_+^{2,\gamma}}$. If J' takes positive values then it is enough to find large c_1 such that

$$\ln \left(\bar{b} \cdot \|r_0\|_{L_+^{2,\gamma}} \right) \leq \ln c_1 - \bar{\lambda} T^* J' \left(\frac{\bar{\lambda} c_1}{\sqrt{\gamma}} \right).$$

Existence of such c_1 is a consequence of (B4). \square

Proof of Theorem 5.5: Since $\bar{h}(\cdot, x)$ is adapted for each $x \geq 0$ as a pointwise limit, we only need to show that $\bar{h}(t)$ is a bounded process in $L_+^{2,\gamma}$, resp. $H_+^{1,\gamma}$. Then \bar{h} solves (4.8) in virtue of Theorem 5.10.

(a) Let c_1 be a constant given by Proposition 8.1. By the Fatou lemma we have

$$\sup_{t \in [0, T^*]} \int_0^{+\infty} |\bar{h}(t, x)|^2 e^{\gamma x} dx \leq \sup_{t \in [0, T^*]} \liminf_{n \rightarrow +\infty} \int_0^{+\infty} |h_n(t, x)|^2 e^{\gamma x} dx \leq c_1^2,$$

and hence $\bar{h}(t)$ is bounded in $L_+^{2,\gamma}$.

(b) In view of (a) we need to show that $h'_x(t)$ is bounded in $L^{2,\gamma}$. Differentiating the equation $\bar{h} = \mathcal{K}\bar{h}$ yields

$$\bar{h}'(t, x) = r'_0(t+x)b(t, x)F_1(t, x) + r_0(t+x)b'_x(t, x)F_1(t, x) + r_0(t+x)b(t, x)F_1(t, x)F_2(t, x),$$

where

$$\begin{aligned} F_1(t, x) &:= e^{\int_0^t J'(\int_0^{t-s+x} \lambda(v)\bar{h}(s, v)dv) \lambda(t-s+x) ds}, \\ F_2(t, x) &:= \int_0^t J'' \left(\int_0^{t-s+x} \lambda(v)\bar{h}(s, v)dv \right) \lambda^2(t-s+x)\bar{h}(s, t-s+x) ds \\ &\quad + \int_0^t J' \left(\int_0^{t-s+x} \lambda(v)\bar{h}(s, v)dv \right) \lambda'_x(t-s+x) ds. \end{aligned}$$

Assumption (A3) implies that $b(\cdot, \cdot)$ and $b'_x(\cdot, \cdot)$ are bounded on $(t, x) \in [0, T^*] \times [0, +\infty)$. Since $r_0 \in H_+^{1,\gamma}$, it is enough to show that

$$\sup_{t \in [0, T^*], x \geq 0} F_1(t, x) < +\infty, \quad \sup_{t \in [0, T^*], x \geq 0} F_2(t, x) < +\infty.$$

We have

$$\sup_{t \in [0, T^*], x \geq 0} F_1(t, x) \leq e^{\left| J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|\bar{h}(t)\|_{L_+^{2,\gamma}} \right) \right| \bar{\lambda} T^*} < +\infty.$$

It follows from Proposition 2.3 that (B4) excludes Wiener part of the noise as well as negative jumps. Thus J'' reduces to the form $J''(z) = \int_0^{+\infty} y^2 e^{-zy} \nu(dy)$ and $0 \leq J''(0) < +\infty$ due to the assumption (B2). Since J'' is decreasing, the following estimation holds

$$\begin{aligned} \sup_{t \in [0, T^*], x \geq 0} F_2(t, x) &\leq J''(0) T^* \bar{\lambda}^2 \sup_{t \in [0, T^*], x \geq 0} \int_0^t \bar{h}(s, t-s+x) ds \\ &\quad + T^* \left| J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|\bar{h}(t)\|_{L_+^{2,\gamma}} \right) \right| \cdot \sup_{x \geq 0} \lambda'(x), \end{aligned}$$

and it is enough to show that \bar{h} is bounded on $\{(t, x), t \in [0, T^*], x \geq 0\}$. In view of the fact that $\bar{h} = \mathcal{K}\bar{h}$ we obtain

$$\sup_{t \in [0, T^*], x \geq 0} \bar{h}(t, x) \leq \sup_{x \geq 0} r_0(x) \cdot \sup_{t \in [0, T^*], x \geq 0} b(t, x) \cdot e^{\left| J' \left(\frac{1}{\sqrt{\gamma}} \sup_t \|\bar{h}(t)\|_{L_+^{2,\gamma}} \right) \right| \bar{\lambda} T^*} < +\infty.$$

□

Proof of Theorem 5.7:

Let r be a solution given by Theorem 5.5(b). Then, by Theorem 5.9, r solves (5.5). We will show that the assumption $\lambda(\cdot) = \lambda$ implies that r is a solution of equation (1.1). Differentiating (5.5) yields

$$\begin{aligned} \frac{\partial}{\partial x} r(t, x) &= e^{\lambda L_t - \frac{q^2 \lambda^2}{2}} \prod (1 + \lambda \Delta L_s) e^{-\lambda \Delta L_s} \\ &\quad \cdot \left(r'_0(t+x) e^{\lambda \int_0^t J'(\lambda \int_0^{t-s+x} r(s, v) dv) ds} + r_0(t+x) e^{\lambda \int_0^t J'(\lambda \int_0^{t-s+x} r(s, v) dv) ds} \right. \\ &\quad \left. \cdot \lambda^2 \int_0^t J'' \left(\lambda \int_0^{t-s+x} r(s, v) dv \right) \cdot r(s, t-s+x) ds \right) \\ &= r(t, x) \frac{r'_0(t+x)}{r_0(t+x)} + r(t, x) \lambda^2 \int_0^t J'' \left(\lambda \int_0^{t-s+x} r(s, v) dv \right) \cdot r(s, t-s+x) ds \\ &= r(t, x) \left[\frac{r'_0(t+x)}{r_0(t+x)} + \lambda^2 \int_0^t J'' \left(\lambda \int_0^{t-s+x} r(s, v) dv \right) \cdot r(s, t-s+x) ds \right]. \end{aligned} \tag{8.6}$$

For Z_1, Z_2 defined by

$$\begin{aligned} Z_1(t) &:= e^{\lambda L_t - \frac{q^2 \lambda^2}{2}} \prod (1 + \lambda \Delta L_s) e^{-\lambda \Delta L_s}, \\ Z_2(t, x) &:= r_0(t+x) e^{\lambda \int_0^t J'(\lambda \int_0^{t-s+x} r(s, v) dv) ds}, \end{aligned}$$

we have SDEs of the form

$$\begin{aligned}
dZ_1(t) &= Z_1(t-)\lambda dL(t) \\
dZ_2(t, x) &= \left\{ r'_0(t+x)e^{\lambda \int_0^t J'(\lambda \int_0^{t-s+x} r(s,v)dv)ds} + r_0(t+x)e^{\lambda \int_0^t J'(\lambda \int_0^{t-s+x} r(s,v)dv)ds} \right. \\
&\quad \cdot \left[\lambda J' \left(\lambda \int_0^x r(t,v)dv \right) + \lambda^2 \int_0^t J'' \left(\lambda \int_0^{t-s+x} r(s,v)dv \right) r(s, t-s+x) ds \right] \Big\} dt \\
&= \left\{ \frac{r'_0(t+x)}{r_0(t+x)} Z_2(t, x) + Z_2(t, x) \left[\lambda J' \left(\lambda \int_0^x r(t,v)dv \right) + \right. \right. \\
&\quad \left. \left. + \lambda^2 \int_0^t J'' \left(\lambda \int_0^{t-s+x} r(s,v)dv \right) r(s, t-s+x) ds \right] \right\} dt \\
&= \left\{ Z_2(t, x) \left[\frac{r'_0(t+x)}{r_0(t+x)} + \lambda J' \left(\lambda \int_0^x r(t,v)dv \right) + \right. \right. \\
&\quad \left. \left. + \lambda^2 \int_0^t J'' \left(\lambda \int_0^{t-s+x} r(s,v)dv \right) r(s, t-s+x) ds \right] \right\} dt.
\end{aligned}$$

Using the formulas above, we obtain SDE for $r(t, x)$:

$$\begin{aligned}
dr(t, x) &= d \left(Z_1(t) Z_2(t, x) \right) = Z_1(t) dZ_2(t, x) + Z_2(t, x) dZ_1(t) \\
&= Z_1(t) Z_2(t, x) \left[\frac{r'_0(t+x)}{r_0(t+x)} + \lambda J' \left(\lambda \int_0^x r(t,v)dv \right) + \right. \\
&\quad \left. + \lambda^2 \int_0^t J'' \left(\lambda \int_0^{t-s+x} r(s,v)dv \right) r(s, t-s+x) ds \right] dt \\
&\quad + Z_2(t, x) Z_1(t-)\lambda dL(t) \\
&= r(t, x) \left[\frac{r'_0(t+x)}{r_0(t+x)} + \lambda^2 \int_0^t J'' \left(\lambda \int_0^{t-s+x} r(s,v)dv \right) r(s, t-s+x) ds \right] dt \\
&\quad + \lambda r(t, x) J' \left(\lambda \int_0^x r(t,v)dv \right) dt + \lambda r(t-, x) dL(t) \\
&\stackrel{\text{by (8.6)}}{=} \frac{\partial}{\partial x} r(t, x) dt + \lambda J' \left(\lambda \int_0^x r(t,v)dv \right) r(t, x) dt + \lambda r(t-, x) dL(t),
\end{aligned}$$

which is (1.1). □

9 Proof of the uniqueness of the solutions in $H_+^{1,\gamma}$

Before presenting the proof of Theorem 5.8 we establish an auxiliary result.

Proposition 9.1 *Let $d : [0, T^*] \times [0, +\infty) \longrightarrow \mathbb{R}_+$ be a bounded function satisfying*

$$d(t, x) \leq C \int_0^t \int_0^{t-s+x} d(s, v) dv ds, \quad (9.1)$$

where $C > 0$ is a fixed constant. Then $d(t, x) = 0$ for all $(t, x) \in [0, T^] \times [0, +\infty)$.*

Proof: Let d be bounded by $M > 0$ on $[0, T^*] \times [0, +\infty)$. Let us define a new function

$$\bar{d}(u, w) := d(u, w - u); \quad u \in [0, T^*], w \geq u.$$

It is clear that $d \equiv 0$ on $[0, T^*] \times [0, +\infty)$ if and only if $\bar{d} \equiv 0$ on the set $\{(u, w) : u \in [0, T^*], w \geq u\}$. Let us notice that (9.1) implies that

$$\begin{aligned} \bar{d}(u, w) &= d(u, w - u) \leq C \int_0^u \int_0^{w-s} d(s, y) dy ds \\ &= C \int_0^u \int_s^w d(s, z - s) dz ds = C \int_0^u \int_s^w \bar{d}(s, z) dz ds. \end{aligned}$$

Using this inequality we will show by induction that

$$\bar{d}(u, w) \leq MC^n \frac{(uw)^n}{(n!)^2}, \quad n = 0, 1, 2, \dots \quad (9.2)$$

Then letting $n \rightarrow 0$ we have $\bar{d}(t, x) = 0$. The formula (9.2) is valid for $n = 0$. Assume that it is true for n and show for $n + 1$:

$$\begin{aligned} \bar{d}(u, w) &\leq C \int_0^u \int_s^w MC^n \frac{(sz)^n}{(n!)^2} dz ds = MC^{n+1} \frac{1}{(n!)^2} \int_0^u s^n \left(\int_s^w z^n dz \right) ds \\ &= MC^{n+1} \frac{1}{(n!)^2} \int_0^u s^n \left(\frac{w^{n+1} - s^{n+1}}{n+1} \right) ds \leq MC^{n+1} \frac{1}{(n!)^2} \int_0^u s^n \frac{w^{n+1}}{n+1} ds \\ &= MC^{n+1} \frac{1}{(n!)^2} \frac{u^{n+1}}{(n+1)} \frac{w^{n+1}}{(n+1)} = MC^{n+1} \frac{(uw)^{n+1}}{((n+1)!)^2}. \end{aligned}$$

□

Proof of Theorem 5.8: Assume that r_1, r_2 are two solutions of the equation (4.8) in $H_+^{1,\gamma}$. Then they are bounded processes in $H^{1,\gamma}$ and, in view of Theorem 5.9, satisfy (5.5). Define

$$d(t, x) := |r_1(t, x) - r_2(t, x)|, \quad 0 \leq t \leq T^*, x \geq 0.$$

Denote $B := \sup_{t \in [0, T^*], x \geq 0} b(t, x)$. The following estimation holds

$$\begin{aligned} d(t, x) &\leq r_0(t+x)b(t, x) \left[e^{\int_0^t J'(\int_0^{t-s+x} \lambda(s,v)r_1(s,v)dv)\lambda(s,t-s+x)ds} + e^{\int_0^t J'(\int_0^{t-s+x} \lambda(s,v)r_2(s,v)dv)\lambda(s,t-s+x)ds} \right] \\ &\leq \sup_{x \geq 0} r_0(x) \cdot B \cdot \left[e^{\bar{\lambda}T^* \left| J'(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|r_1(t)\|_{L_+^{2,\gamma}}) \right|} + e^{\bar{\lambda}T^* \left| J'(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|r_2(t)\|_{L_+^{2,\gamma}}) \right|} \right] < +\infty, \end{aligned}$$

and thus d is bounded on $[0, T^*] \times [0, +\infty)$. In view of the inequality $|e^x - e^y| \leq e^{x \vee y} |x - y|$; $x, y \geq 0$ and the fact that J'' is decreasing with $0 \leq J''(0) < +\infty$ due to assumption (B2), we

have

$$\begin{aligned}
d(t, x) &\leq \sup_{x \geq 0} r_0(x) \cdot Be^{\max \left\{ \int_0^t J' \left(\int_0^{t-s+x} \lambda(s, v) r_1(s, v) dv \right) \lambda(s, t-s+x) ds; \int_0^t J' \left(\int_0^{t-s+x} \lambda(s, v) r_2(s, v) dv \right) \lambda(s, t-s+x) ds \right\}} \\
&\cdot \left| \int_0^t J' \left(\int_0^{t-s+x} \lambda(s, v) r_1(s, v) dv \right) \lambda(s, t-s+x) ds - \int_0^t J' \left(\int_0^{t-s+x} \lambda(s, v) r_2(s, v) dv \right) \lambda(s, t-s+x) ds \right| \\
&\leq \sup_{x \geq 0} r_0(x) \cdot Be^{\bar{\lambda} T^* \max \left\{ \left| J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|r_1(t)\|_{L^2_+, \gamma} \right) \right|; \left| J' \left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \sup_t \|r_2(t)\|_{L^2_+, \gamma} \right) \right| \right\}} \\
&\cdot J''(0) \bar{\lambda}^2 \int_0^t \int_0^{t-s+x} |r_1(s, v) - r_2(s, v)| dv ds = C \int_0^t \int_0^{t-s+x} d(s, v) dv ds, \quad (t, x) \in [0, T^*] \times [0, +\infty).
\end{aligned}$$

It follows from Proposition 9.1 that $r_1 = r_2$ on $[0, T^*] \times [0, +\infty)$. \square

10 Appendix

10.1 HJM approach to the bond market

Let $P(t, T)$ denote a price at time $t \geq 0$ of a bond paying 1 unit of money to its holder at time $T \geq t$. The prices $P(\cdot, T)$ are processes defined on a fixed filtered probability space $(\Omega, \mathcal{F}_{t, t \geq 0}, P)$. The forward rate f is a random field defined by the formula

$$P(t, T) = e^{-\int_t^T f(t, u) du}, \quad 0 \leq t \leq T \leq T^*.$$

The prices of all bonds traded on the market are thus determined by the forward rate $f(t, T)$, $0 \leq t \leq T < +\infty$ and thus the starting point in the bond market description is specifying the dynamics of f . In this paper we consider the following stochastic differentials

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dL(t), \quad 0 \leq t \leq T, \quad (10.1)$$

where L is a Lévy process. The equation above can be viewed as a system of infinitely many equations parameterized by $0 \leq T < +\infty$. The discounted bond prices $\hat{P}(t, T)$ are defined by

$$\hat{P}(t, T) := e^{-\int_0^t v(s) ds} \cdot P(t, T), \quad 0 \leq t \leq T < +\infty,$$

where $v(t) := f(t, t)$, $t \geq 0$ is the short rate. If we extend the domain of f by putting $f(t, T) = f(T, T)$ for $t \geq T$ we obtain the formula

$$\hat{P}(t, T) = e^{-\int_0^T f(t, u) du}, \quad 0 \leq t \leq T < +\infty.$$

The market is supposed to be arbitrage free, i.e. we assume that the processes $\hat{P}(\cdot, T)$ are local martingales. This implies that the coefficients α, σ in (10.1) satisfy the Heath-Jarrow-Morton condition, i.e. for each $T \geq 0$

$$\int_t^T \alpha(t, u) du = J \left(\int_t^T \sigma(t, u) du \right), \quad (10.2)$$

for almost all $t \geq 0$, see [3], [4], [9]. The function J above is the Laplace exponent of L defined by (2.1). As J is differentiable, (10.2) can be written as

$$\alpha(t, T) = J' \left(\int_t^T \sigma(t, u) du \right) \sigma(t, T), \quad 0 \leq t \leq T < +\infty,$$

which means that the drift is fully determined by the volatility process. As a consequence (10.1) reads as

$$f(t, T) = f(0, T) + \int_0^t J' \left(\int_s^T \sigma(s, u) du \right) \sigma(s, T) ds + \int_0^t \sigma(s, T) dL(s), \quad 0 \leq t \leq T < +\infty. \quad (10.3)$$

If we put $x = T - t$ then from the above we obtain (2.7) for the dynamics of $r(t, x)$, which is a weak form of (1.1).

The assumptions that the process $r(t, \cdot), t \geq 0$, takes values in $L^{2, \gamma}$ or in $H^{1, \gamma}$ have financial interpretations. For instance if $r \in L^{2, \gamma}$ then

$$\begin{aligned} \int_0^{+\infty} |r(x)| dx &= \int_0^{+\infty} |r(x)| e^{\frac{\gamma}{2}x} \cdot e^{-\frac{\gamma}{2}x} dx \leq \left(\int_0^{+\infty} |r(x)|^2 e^{\gamma x} dx \right)^{\frac{1}{2}} \left(\int_0^{+\infty} e^{-\gamma x} dx \right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{\gamma}} \|r\|_{L^{2, \gamma}} < +\infty. \end{aligned} \quad (10.4)$$

Consequently, for fixed t and all $T \geq t$,

$$P(t, T) \geq e^{-\frac{1}{\sqrt{\gamma}} \|r(t)\|_{L^{2, \gamma}}},$$

and therefore the bond prices, as functions of the maturity T , are bounded from below by a positive number. The requirement

$$\int_0^{+\infty} |r'_x(t, x)|^2 e^{\gamma x} dx < +\infty$$

corresponds to the observation that the forward rates are getting flat for large maturities T .

10.2 Laplace exponent

To examine properties of the Laplace exponent

$$J(z) = -az + \frac{1}{2}qz^2 + \int_{-\infty}^{+\infty} (e^{-zy} - 1 + zy\mathbf{1}_{(-1,1)}(y)) \nu(dy), \quad z \in \mathbb{R},$$

let us represent it in the form

$$J(z) = -az + \frac{1}{2}qz^2 + J_1(z) + J_2(z) + J_3(z) + J_4(z),$$

where

$$\begin{aligned} J_1(z) &:= \int_{-\infty}^{-1} (e^{-zy} - 1) \nu(dy), & J_2(z) &:= \int_{-1}^0 (e^{-zy} - 1 + zy) \nu(dy), \\ J_3(z) &:= \int_0^1 (e^{-zy} - 1 + zy) \nu(dy), & J_4(z) &:= \int_1^{+\infty} (e^{-zy} - 1) \nu(dy). \end{aligned}$$

If the integrals below exist then we have the following formulas for the derivatives, see for instance Lemma 8.1 and 8.2 in [17]

$$\begin{aligned}
J_1'(z) &:= - \int_{-\infty}^{-1} y e^{-zy} \nu(dy), & J_2'(z) &:= \int_{-1}^0 y (1 - e^{-zy}) \nu(dy), \\
J_3'(z) &:= \int_0^1 y (1 - e^{-zy}) \nu(dy), & J_4'(z) &:= - \int_1^{+\infty} y e^{-zy} \nu(dy); \\
J_1''(z) &:= \int_{-\infty}^{-1} y^2 e^{-zy} \nu(dy), & J_2''(z) &:= \int_{-1}^0 y^2 e^{-zy} \nu(dy), \\
J_3''(z) &:= \int_0^1 y^2 e^{-zy} \nu(dy), & J_4''(z) &:= \int_1^{+\infty} y^2 e^{-zy} \nu(dy); \\
J_1'''(z) &:= - \int_{-\infty}^{-1} y^3 e^{-zy} \nu(dy), & J_2'''(z) &:= - \int_{-1}^0 y^3 e^{-zy} \nu(dy), \\
J_3'''(z) &:= - \int_0^1 y^3 e^{-zy} \nu(dy), & J_4'''(z) &:= - \int_1^{+\infty} y^3 e^{-zy} \nu(dy).
\end{aligned}$$

Below we gather properties of J needed in the paper. The domain of J is restricted to the half-line $[0, +\infty)$ due to the fact that we are interested in positive solutions of (1.1) only. For $z > 0$, $|J'(z)| < +\infty$ if $J_1'(z)$ is well defined, that is if

$$\int_{-\infty}^{-1} |y| e^{z|y|} \nu(dy) < +\infty.$$

Moreover,

- $|J'(0)| < +\infty$ iff

$$(B0) \quad \int_{|y|>1} |y| \nu(dy) < +\infty, \tag{10.5}$$

and

- J' is increasing.

Moreover, it follows from the below formulas

$$\lim_{z \rightarrow +\infty} |J_1'(z)| = +\infty, \quad \lim_{z \rightarrow +\infty} |J_2'(z)| = +\infty,$$

$$|J_3'| \text{ is bounded} \iff \int_0^1 y \nu(dy) < +\infty, \quad |J_4'(z)| \text{ is bounded} \iff \int_1^{+\infty} y \nu(dy) < +\infty,$$

that under (10.5)

- J' is bounded on $[0, +\infty)$ iff

$$(B1) \quad \left\{ \begin{array}{l} \bullet \quad L \text{ does not have the Wiener part, i.e. } q = 0, \\ \bullet \quad \text{supp}\{\nu\} \subseteq [0, +\infty), \\ \bullet \quad \int_0^{+\infty} y \nu(dy) < +\infty. \end{array} \right.$$

By similar analysis

$$\lim_{z \rightarrow +\infty} |J_1''(z)| = +\infty, \quad \lim_{z \rightarrow +\infty} |J_2''(z)| = +\infty,$$

$$|J_3''| \text{ is bounded, } |J_4''(z)| \text{ is bounded} \iff \int_1^{+\infty} y^2 \nu(dy) < +\infty,$$

we conclude that

- J'' is bounded on $[0, +\infty)$ iff

$$(B2) \quad \left\{ \begin{array}{l} \bullet \quad \text{supp}\{\nu\} \subseteq [0, +\infty), \\ \bullet \quad \int_1^{+\infty} y^2 \nu(dy) < +\infty. \end{array} \right.$$

J' is bounded on $[0, z_0]$, $z_0 > 0$ iff $\left| \int_{|y|>1} y e^{-z_0 y} \nu(dy) \right| < +\infty$ and is finite at 0. Thus

- J' is locally bounded iff (10.5) holds and $\int_{y<-1} |y| e^{z_0 |y|} \nu(dy) < +\infty$.

Similarly,

- J' is locally Lipschitz iff

$$(L1) \quad \left\{ \begin{array}{l} \bullet \quad \int_{-\infty}^{-1} |y|^2 e^{z_0 |y|} \nu(dy) < +\infty, \\ \bullet \quad \int_{|y|>1} |y|^2 \nu(dy) < +\infty, \end{array} \right.$$

and

- J'' is locally Lipschitz iff

$$(L2) \quad \left\{ \begin{array}{l} \bullet \quad \int_{-\infty}^{-1} |y|^3 e^{z_0 |y|} \nu(dy) < +\infty, \\ \bullet \quad \int_{|y|>1} |y|^3 \nu(dy) < +\infty. \end{array} \right.$$

For the linear case we assume that the support of the Lévy measure is contained in $[-\frac{1}{\bar{\lambda}}, +\infty)$, where $-\infty < \bar{\lambda} < +\infty$. Thus the above results can be written in the simpler form. The assumption (10.5) reduces to the form

-

$$|J'(0)| < +\infty \iff \int_{y>1} y \nu(dy) < +\infty, \quad (10.6)$$

and

- J' is locally bounded \iff (10.6) holds,
- J' is locally Lipschitz $\iff \int_1^{+\infty} y^2 \nu(dy) < +\infty$,
- J'' is locally Lipschitz $\iff \int_1^{+\infty} y^3 \nu(dy) < +\infty$.

References

- [1] Barski M., Zabczyk J.: "Forward rate models with linear volatility", (2012), *Finance and Stochastics*, 16,3, 537-560,
- [2] Barski M., Zabczyk J.: "Heath-Jarrow-Morton-Musiela equation with linear volatility", (2011) <http://arxiv.org/abs/1010.5808>,
- [3] Björk, Th., Di Masi, G., Kabanov, Y., Runggaldier, W.: "Towards a general theory of bond markets", (1997), *Finance and Stochastics* 1, 141-174,
- [4] Eberlein, E., Raible, S.: "Term structure models driven by general Lévy processes", (1999), *Math. Finance*, 9, 31-53,
- [5] Filipović D., Tappe S.: "Existence of Lévy term structure models", (2008), *Finance and Stochastic*, 12, 83-115,
- [6] Filipović, D.: "Consistency problems for Heath-Jarrow-Morton interest rate models", (2001), *Lecture Notes in Mathematics*, vol.1760,
- [7] Filipović, D.: "Term-Structure Models: A Graduate Course", (2009), Springer-Verlag,
- [8] Filipović, D., Tappe, S., Teichmann, J. : "Term structure models driven by Wiener process and Poisson measures: Existence and positivity", (2010), *SIAM Journal on Financial Mathematics*, Vol.1, 523-554,
- [9] Jakubowski, J., Zabczyk J.: "Exponential moments for HJM models with jumps", (2007), *Finance and Stochastics*, 11, 429-445,
- [10] Marinelli, C.: "Local well-posedness of Musiela's SPDE with Lévy noise", (2010), *Mathematical Finance*, 20, 341-363,
- [11] Milian, A. : "Comparison theorems for stochastic evolution equation", (2002), *Stochastics and Stochastics Reports*, 72, 79-108,
- [12] Morton, A.: "Arbitrage and martingales", (1989), Dissertation, Cornell University,
- [13] Musiela, M. "Stochastic PDEs and term structure models", (1993), *Journées International de Finance, IGR-AFFI, La Baule*.

- [14] Peszat, Sz., Zabczyk J.: "Stochastic partial differential equations with Lévy noise", (2007), Cambridge University Press,
- [15] Peszat, Sz., Zabczyk, J.: "Heath-Jarrow-Morton-Musiela equation of bond market", (2007), <http://www.impan.pl/Preprints/p677.pdf>,
- [16] Protter, P.: "Stochastic Integration and Differential Equations", (2005), *Springer, Berlin*.
- [17] Rusinek, A.: "Invariant measures for forward rate HJM model with Lévy noise", (2006), Preprint IMPAN 669, <http://www.impan.pl/Preprints/p669.pdf>,
- [18] Rusinek, A.: "Properties of solutions to the Musiela equation with Lévy noise", (2011), *PhD Dissertation, Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland*.
- [19] Rusinek, A.: "Mean reversion for HJMM forward rate models", (2010), *Adv. in Appl. Probab.*, 42, 371-391.